This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

[redacted]

Per 6440

[redacted]

Harvard College Library

FROM THE BEQUEST OF

HORACE APPLETON HAVEN,

OF PORTSMOUTH, N. H.

(Class of 1842.)

SCIENCE CENTER LIBRARY

[redacted]

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME VI.

SESSION 1887–88.

EDINBURGH:
1888.

Digitized by Google

Sc 890.90
(C IX 63)

PER
6440

HARVARD COLLEGE
MAR 2 1911

Haven fund

PRINTED BY

JOHN LINDSAY, HIGH STREET, EDINBURGH.

Digitized by Google

INDEX.

PAGE

ALLARDICE, R. E.

A Method of Transformation in Geometry, . . . 14
On Stirling's Approximation to $n!$ when n is large, . 22
On the Inscription of a Triangle of Given Shape in a Given Triangle, ... 42
A Construction for the Brocard Points, ... 88

AUBERT, PAUL

Sur un Système de Cercles Tangents à une Circonférence et Orthogonaux à une autre Circonférence, . . 33

CHREE, CHARLES

"Vortex Rings in a Compressible Fluid," . . . 59

CHRYSTAL, Professor

On the Inequality $mx^{m-1}(x-1) \gtrless x^m - 1 \gtrless m(x-1)$ and its Consequences, ... 29

ELLIOTT, A. C.

The Potential of a Spherical Magnetic Shell deduced from the Potential of a Coincident Layer of Attracting Matter, ... 12

FRASER, A. Y.

A Device for the Analysis of Intervals and Chords in Music, ... 24

Digitized by Google

PAGE

GIBSON, GEORGE A.

Extension of a Theorem of Abel's in Summation to Integration, ... 40

MACKAY, J. S.

Properties of the Figure consisting of a Triangle, and the Squares described on its Sides, ... 2

Pappus on the Progressions, ... 48

Similitude and Inversion, ... 69

OFFICE-BEARERS, ... 1

PEDDIE, W.

On the Thermodynanic Surface [Abstract], ... 33

REPORT ON THE TEACHING OF ARITHMETIC, ... 89

Digitized by Google

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

SIXTH SESSION, 1887-88.

First Meeting, November 11th, 1887.

W. J. MACDONALD, ESQ., M.A., F.R.S.E., Vice-President, in the Chair.

Dr GEORGE THOM, as retiring President, delivered an Address on "Mathematics in Secondary Schools and Universities."

For this Session the following Office-Bearers were elected :—

President, . .	Mr W. J. MACDONALD, M.A., F.R.S.E.
Vice-President, . . .	Mr GEORGE A. GIBSON, M.A.
Secretary, . . .	Mr A. Y. FRASER, M.A., F.R.S.E.
Treasurer, ...	Mr JOHN ALISON, M.A.

Committee.

Mr R. E. ALLARDICE, M.A.	Mr W. PEDDIE, B.Sc., F.R.S.E.
Mr ARCHD. C. ELLIOTT, B.Sc.	Dr GEORGE THOM.
Mr F. GRANT OGILVIE, M.A. B.Sc.	Rev. JOHN WILSON, M.A., F.R.S.E.

Of the above, Messrs ALLARDICE *and* PEDDIE *were elected to edit the Proceedings.*

Digitized by Google

Properties of the figure consisting of a triangle, and the squares described on its sides.

By J. S. Mackay, LL.D.

[Of the following properties, some must have been long known, but I do not remember any early statement of them. The 1st, 10th, 15th, 16th, 19th, 20th, and 22nd are given, mostly without proof, in an article by Vecten in Gergonne's *Annales de Mathématiques*, vol. VII., p. 321 (1817); the first parts of the 3rd and 4th are proposed for proof by William Godward in *The Gentleman's Diary* for 1837, p. 48, and proved by him and others in the *Diary* for 1838, pp. 40–41; the first parts of the 5th, 6th, 12th, and 14th are given in M'Dowell's *Exercises on Euclid and in Modern Geometry*, §§ 27, 28, 29 (1863); the 9th in Milne's *Weekly Problem Papers*, p. 135 (1885); the first part of the 8th in Vuibert's *Journal de Mathématiques Elémentaires*, 12e Année, p. 18 (1887); the 39th in the *Journal de Mathématiques Elémentaires*, edited by De Longchamps, 3e série, tome I., p. 234 (1887). The others are believed to be new.

For the sake of brevity, I have given only one of the varieties of the figure—that, namely, where the squares are all described outwardly on the sides of the triangle, and I have not mentioned any of the numerous properties that may be derived from the several sets of concurrent straight lines.]

Figure 1.

Let ABC be a triangle; BDEC, CFGA, AHIB the squares described outwardly on its sides. GH, ID, EF are joined, and on GH is described the square GHKL; IK and FL are joined. Then

§ 1. The triangles GAH, IBD, ECF are each equivalent to the triangle ABC.

Since angles BAH, CAG are right, therefore angle GAH is supplementary to angle CAB; and GA, AH are respectively equal to CA, AB; therefore triangle GAH is equal to triangle CAB.

§ 2. The triangles FGL, IHK are each equivalent to the triangle ABC.

For these triangles are each equivalent to the triangle GAH, and therefore to the triangle ABC.

Digitized by Google

Figure 2.

§ 3. The perpendicular from A to GH is the median from A to BC, and the perpendicular from A to BC is the median from A to GH.

Let AY which is perpendicular to GH meet BC at M, and from B and C draw BP, CQ perpendicular to AM.

Then the right-angled triangles AYH, BPA, having angles HAY, BAP complementary, and AH = BA, are congruent;
therefore AY = BP.

Similarly from the right-angled triangles AYG, CQA there results AY = CQ; therefore BP = CQ.

Hence the right-angled triangles BMP, CMQ are congruent, and BM = CM.

From the relation in which the triangles ABC, AHG stand to each other, it follows from the preceding reasoning that if AX, which is perpendicular to BC, meet HG at N, then HN = GN.

Figure 1.

§ 4. The perpendiculars from A to GH, from B to ID, from C to EF intersect at the centroid of the triangle ABC; and the medians from A to GH, from B to ID, from C to EF intersect at the orthocentre of the triangle ABC.

This follows at once from the preceding section.

Figure 2.

§ 5. GH is double of the median from A to BC, and BC is double of the median from A to GH.

The congruent triangles AYG, CQA give GY = AQ;
and the congruent triangles AYH, BPA give HY = AP;
therefore GH = GY + HY = AQ + AP.
But the congruent triangles BMP, CMQ give MP = MQ;
therefore AQ + AP = 2AM; therefore GH = 2AM.

Hence also, as in § 3, BC = 2AN.

Figure 1.

§ 6. The triangle whose sides are GH, ID, EF is three times the triangle ABC;
and the triangle whose sides are BC, FL, KI is three times the triangle ABC.

Digitized by Google

Slide the triangle IBD, without rotating it, in such a manner that B may move along BA to A; then BI will coincide with AH. Similarly slide the triangle FCE so that C may move along CA to A; then CF will coincide with AG. If it be not clear that BD and CE will coincide after these transferences, it may be noted that the sum of the angles GAH, IBD, ECF is four right angles.

Hence the three triangles GAH, IBD, ECF, each of which is equal to the triangle ABC, form a single triangle whose sides are GH, ID, EF.

From the relation in which the triangles ABC, AHG stand to each other, it follows that BC, FL, KI will form sides of a triangle which is three times the triangle AHG, and consequently three times the triangle ABC.

Figure 3.

§ 7. The sides of the triangle ABC are two-thirds of the medians of the triangle whose sides are GH, ID, EF.

Let GHD represent the triangle formed by the junction round the point A of the three equal triangles GAH, IBD, ECF. Then AH, AD, AG will represent AB, BC, CA.

Since the triangles GAH, HAD, DAG are all equal, therefore A is the centroid of the triangle GHD, and HA, DA, GA produced are the medians. Now HA, DA, GA, and consequently AB, BC, CA, are two-thirds of the medians from H, D, G.

§ 8. From the results established in the three preceding sections may be derived some of the relations which exist between a triangle and the triangle whose sides are the medians of the former.

(*a*) Let ABC, A′B′C′, ... DEF, D′E′F′ ... be two sets of triangles, such that the sides of DEF are the medians of ABC, the sides of A′B′C′ are the medians of DEF, the sides of D′E′F′ are the medians of A′B′C′, and so on; the set of triangles ABC, A′B′C′, ... will be similar to each other, and the set DEF, D′E′F′ ... similar to each other.

To prove this, consider the triangle whose sides are GH, ID, EF, and the triangle ABC.

Figure 1.

The triangle ABC has its sides two-thirds of the medians of the triangle whose sides are GH, ID, EF; and therefore it is similar to the triangle whose sides are the medians.

Digitized by Google

But the medians of the triangle ABC are respectively halves of GH, ID, EF; therefore the triangle whose sides are the medians of ABC is similar to the triangle whose sides are GH, ID, EF.

(*b*) Let ABC be a triangle whose centroid is G, DEF the triangle whose sides are the medians of ABC, the angles of DEF are such that

$$D = GCA + GAC,\ E = GAB + GBA,\ F = GBC + GCB.$$

For (*Figure* 1) the angle CAB = AGH + AHG, angle ABC = BID + BDI, angle BCA = CEF + CFE. But AGH, AHG, BID, BDI, CEF, CFE are the six angles formed by the medians at the vertices of the triangle whose sides are GH, ID, EF, and the triangle ABC is similar to the triangle formed by these medians.

(*c*) Let ABC be a triangle, DEF the triangle whose sides are the medians of ABC, then DEF = $\frac{3}{4}$ABC.

For (*Figure* 1) the sides of triangle ABC are two-thirds of the medians of the triangle whose sides are GH, ID, EF; therefore the area of triangle ABC is four-ninths of the area of the triangle formed by these medians. But the triangle whose sides are GH, ID, EF is three times the triangle ABC; therefore the triangle whose sides are GH, ID, EF is $(4/9) \times 3$ times the triangle formed by its medians.

§ 9. If in the same way that the triangle whose sides are GH, ID, EF is formed from ABC, there be formed a third triangle from that whose sides are GH, ID, EF, the sides of this third triangle will be three times the sides of ABC.

For this third triangle will be three times the triangle whose sides are GH, ID, EF, and therefore nine times the triangle ABC; and it is similar to ABC. Hence its sides must be three times the sides of ABC.

§ 10. $GH^2 + ID^2 + EF^2 = 3(AB^2 + BC^2 + CA^2)$.

Rotate the triangle GAH round A counter-clockwise through a right angle; then AH will coincide with AB, and AG will be CA produced. Do the same with the triangles IBD and ECF round B and C; and there will be obtained *Figure* 4 in which AG = CA, BI = AB, CE = BC, and where GB, IC, EA represent GH, ID, EF in *Figure* 1.

Now
$$GB^2 + BC^2 = 2CA^2 + 2AB^2,$$
$$IC^2 + CA^2 = 2AB^2 + 2BC^2,$$
$$EA^2 + AB^2 = 2BC^2 + 2CA^2;$$
therefore
$$GB^2 + IC^2 + EA^2 = 3(AB^2 + BC^2 + CA^2).$$

Digitized by Google

[Perhaps the following proof may be thought simpler :—

Since BA and AC are respectively equal to HA and AG, and since the defect of angle BAC from a right angle equals the excess of angle HAG above a right angle, therefore BC^2 is as much less than $BA^2 + AC^2$ as HG^2 is greater than $HA^2 + AG^2$;

therefore $BC^2 + GH^2 = 2(AB^2 + CA^2)$, and so on.]

§ 11. If as the square GHKL is described on GH, squares be described on ID and EF, and the two pairs of straight lines that correspond to FK, IL be drawn, the sum of the squares on these six straight lines is equal to eight times the sum of the squares on the sides of ABC.

For $BC^2 + FL^2 + KI^2 = 3(AG^2 + GH^2 + HA^2)$

$= 3(CA^2 + GH^2 + AB^2)$,

therefore $FL^2 + KI^2 = 3(CA^2 + AB^2 + GH^2) - BC^2$.

Hence the sum of the squares on the six straight lines

$= 3(CA^2 + AB^2 + GH^2) - BC^2$

$+ 3(AB^2 + BC^2 + ID^2) - CA^2$

$+ 3(BC^2 + CA^2 + EF^2) - AB^2$

$= 5(AB^2 + BC^2 + CA^2) + (GH^2 + ID^2 + EF^2)$,

$= 8(AB^2 + BC^2 + CA^2)$.

Figure 1.

§ 12. The projections of BI and CF on BC are each equal to the perpendicular from A on BC; and the projections of HI and GF on GH are each equal to the perpendicular from A on GH.

Let I', F' be the projections of I, F on BC.

From the congruency of the triangles BII', ABX

it follows that $BI' = AX$;

and from the congruency of the triangles CFF', ACX

it follows that $CF' = AX$.

Similarly if I'', F'' be the projections of I, F on GH,

it follows that $HI'' = AY$, and $GF'' = AY$.

§ 13. M is the middle point of I'F', and N of I''F''.

This follows at once from the preceding section.

§ 14. The sum of the perpendiculars from I and F on BC is equal to BC; and the sum of the perpendiculars from I and F on GH is equal to GH.

From the congruency of the triangles BII', ABX

it follows that $II' = BX$;

Digitized by Google

and from the congruency of the triangles CFF', ACX
it follows that $FF' = CX$;

therefore $II' + FF' = BX + CX = BC$.

If I and F be on opposite sides of BC, the difference of II' and FF' must be taken instead of the sum.

Similarly $II'' + FF'' = GH$.

§ 15. BG, CH are equal and mutually perpendicular; so are CI, AD; and AE, BF.

For in the triangles BAG, HAC, $BA = HA$, $AG = AC$, and angle BAG = angle HAC; therefore $BG = HC$.

Again if the triangle BAG be rotated clockwise through a right angle, BA will coincide with HA, AG with AC, and consequently BG with HC.

BG therefore in its first position is perpendicular to HC.

The point of intersection of BG, CH is P_1; of CI, AD is P_2; of AE, BF is P_3.

§ 16. BF, CI intersect on the perpendicular from A to BC; CH, AE on the perpendicular from B to CA; and AD, BG on the perpendicular from C to AB.

Slide the triangle ADE, without rotating it, in such a manner that A may move along XA produced to a distance equal to BC; and let T_1 be the new position of A. Then D and E will coincide with B and C; and since T_1A is equal and parallel to BD and CE, therefore T_1B is parallel to AD, and T_1C parallel to AE. Now since BF is perpendicular to AE, and CI to AD, therefore BF is perpendicular to T_1C, and CI to T_1B. Hence T_1X, BF, CI are the three perpendiculars of the triangle T_1BC, and therefore intersect at one point.

The point of intersection of BF, CI is Q_1; of CH, AE is Q_2; of AD, BG is Q_3.

§ 17. The distance of Q_1 from BC is a fourth proportional to $BC + AX$, CX, BX.

For the similar triangles BFF', BQ_1X give

$$BF' : FF' = BX : Q_1X,$$

that is
$$BC + AX : CX = BX : Q_1X.$$

Similar statements hold for Q_2 and Q_3.

§ 18. AT_1 is bisected by GH,

Digitized by Google

Because AT_1 is perpendicular to BC, therefore it coincides with the median from A to GH, that is, it passes through N. But BC was proved in § 5 to be double of AN; therefore AT_1 is double of AN.

§ 19. The points P_1, P_2 are on the circle circumscribed about AHIB; P_2, P_3 on the circle BDEC; and P_3, P_1 on the circle CFGA.

Because the angle BP_1H is right, therefore P_1 lies on the circle whose diameter is BH, that is on the circle AHIB; and because the angle AP_2I is right, therefore P_2 lies on the circle whose diameter is AI, that is on the circle AHIB.

§ 20. FI, HE, DG pass respectively through the points P_1, P_2, P_3, and bisect the angles formed by BG and CH, by CI and AD, by AE and BF.

Join FP_1, IP_1.

Because P_1 is a point on the circumference of the circle CFGA, and CF is the chord of a quadrant of that circle, therefore the angle CP_1F is half a right angle. For a similar reason the angle HP_1I is half a right angle; therefore, since CP_1H is a straight line, FP_1I is also a straight line.

Again, the angles CP_1F, GP_1F are each half a right angle; therefore FI bisects the angles formed by BG and CH.

§ 21. AP_1 is perpendicular to FI, BP_2 to HE, and CP_3 to DG.

For the angles AP_1H, HP_1I at the circumference of the circle AHIB stand on arcs each equal to a quadrant.

§ 22. AP_1, BP_2, CP_3 are concurrent at R.

For AP_1 is the common chord or radical axis of the circles AHIB, CFGA; BP_2 that of the circles AHIB, BDEC; CP_3 that of the circles BDEC, CFGA. Hence AP_1, BP_2, CP_3 are concurrent, and R is the radical centre of the three circles.

§ 23. If U, V, W are the centres of the squares described on BC, CA, AB, then VW is parallel to FI and equal to half of it; and similar relations hold between WU and HE, and UV and DG.

For in the triangle AFI, VW joins the middle points of the sides AF, AI; therefore VW is parallel to FI and equal to half of it.

§ 24. AP_1, BP_2, CP_3 pass respectively through U, V, W, and R is the orthocentre of the triangle UVW.

Digitized by Google

For the angle BP_1C is right, and AP_1 bisects it; therefore AP_1 passes through the centre of the square described on BC.*

Hence UA, VB, WC are the three perpendiculars of the triangle UVW, and R is its orthocentre.

§ 25. If U′ be the centre of the square described on GH, then U, A, U′ are collinear.

For the point P_1 stands in the same relation to the triangle AHG as it does to the triangle ABC, and consequently AP_1 passes also through U′.

§ 26. WU passes through the intersection of CI and BT_1; and UV through the intersection of BF and CT_1.

Since the angles BP_2D, BP_2I are each half a right angle, and since BT_1 is perpendicular to CI, therefore BP_2 is one of the diagonals of a square, three of whose sides coincide with BT_1, CI, AD. Hence the other diagonal of this square, which passes through the intersection of CI and BT_1, will bisect BP_2 perpendicularly. But BP_2 is the common chord of the two circles AHIB, BDEC; therefore it is bisected perpendicularly by WU, the straight line joining the centres. Hence WU passes through the intersection of CI and BT_1.

§ 27. If VW be the side of a square, BG or CH is its diagonal; and a similar relation holds between WU and CI or AD, and between UV and AE or BF.

For $BA : WA = \sqrt{2} : 1$, and $AG : AV = \sqrt{2} : 1$;
therefore in the triangles BAG, WAV, $BA : AG = WA : AV$, and the angle BAG = the angle WAV, because each is equal to the angle BAC increased by a right angle. Hence these triangles are similar, and $BG : WV = \sqrt{2} : 1$.

§ 28. If BG or CH be the side of a square, FI is its diagonal; and a similar relation holds between CI or AD and HE, and between AE or BF and DG.

For $FI : VW = 2 : 1$, and $BG : VW = \sqrt{2} : 1$;
therefore $FI : BG = \sqrt{2} : 1$.

§ 29. The result obtained in the preceding paragraph and several other results may be got in the following way.

Since the points P_1, P_2, P_3 are situated in pairs on the circum-

* See (9) of Mr William Harvey's *Notes on Euclid*, I. 47, in the Proceedings of the Edinburgh Mathematical Society, vol. IV., p. 19.

Digitized by Google

ferences of the circles circumscribed about the squares, twelve encyclic quadrilaterals are formed, AP_1IH, AP_2IH, BP_1HI, BP_2HI, and two other sets of four corresponding to the sides BC, CA. From two of them BP_1HI and CP_1GF, by the application of Ptolemy's theorem, there result

$$BP_1 \cdot HI + HP_1 \cdot BI = IP_1 \cdot BH,$$

and $$CP_1 \cdot GF + GP_1 \cdot CF = FP_1 \cdot CG.$$

Now $HI = BI = BH/\sqrt{2}$, and $GF = CF = CG/\sqrt{2}$;

therefore $$BP_1 + HP_1 = IP_1\sqrt{2}, \qquad (\alpha)$$

and $$CP_1 + GP_1 = FP_1\sqrt{2}; \qquad (\beta)$$

therefore $BG + CH = FI\sqrt{2}$, by addition;

therefore $2BG = 2CH = FI\sqrt{2}$

therefore $FI : BG = \sqrt{2} : 1.$

§ 30. From the equations (α) and (β), by subtraction, there results

$$(BP_1 - CP_1) + (HP_1 - GP_1) = (IP_1 - FP_1)\sqrt{2}.$$

Now $BP_1 - CP_1 = HP_1 - GP_1$, since $BP_1 + GP_1 = CP_1 + HP_1$;

therefore $2(BP_1 - CP_1) = 2(HP_1 - GP_1) = (IP_1 - FP_1)\sqrt{2}$,

or $IP_1 - FP_1 = (BP_1 - CP_1)\sqrt{2} = (HP_1 - GP_1)\sqrt{2}$.

Similar relations hold for $EP_2 - HP_2$ and $GP_2 - DP_2$.

§ 31. From the encyclic quadrilaterals AP_1IH and AP_1FG there result

$$AP_1 + IP_1 = HP_1\sqrt{2},$$

and $$AP_1 + FP_1 = GP_1\sqrt{2};$$

therefore $2AP_1 + FI = (HP_1 + GP_1)\sqrt{2}$, by addition.

Similar relations hold for $2BP_2 + HE$, and $2CP_2 + DG$.

§ 32. $AU = VW$, $BV = WU$, $CW = UV$.

Because $(GP_1 + HP_1)\sqrt{2} = 2AP_1 + FI$,

therefore $GP_1 + HP_1 = AP_1\sqrt{2} + FI/\sqrt{2}$,

$= AP_1\sqrt{2} + BG.$

Now $BP_1 + CP_1 = UP_1\sqrt{2}$*;

therefore $BG + CH = AU\sqrt{2} + BG$, by addition;

therefore $CH = AU\sqrt{2}$.

But $CH = VW\sqrt{2}$; therefore $AU = VW$.

---

* See (5) of Mr Harvey's paper, before referred to.

Digitized by Google

This result will be established more simply later on.

§ 33. The perpendiculars from M and N to BC and GH intersect each other at the middle point of FI.

Because M is the middle point of F'I', and FF', II' are perpendicular to BC, therefore the perpendicular to BC at M will bisect FI. For a similar reason the perpendicular to GH at N will bisect FI.

§ 34. If Z be the middle point of FI, then ZBUC and ZHU'G are squares.

For $ZM = \frac{1}{2}(FF' + II') = \frac{1}{2}BC$; and UM is perpendicular to BC and equal to $\frac{1}{2}BC$; therefore ZBUC is a square.

For a similar reason ZHU'G is a square.

§ 35. AMZN is a parallelogram whose diagonals intersect at the middle point of VW.

For $ZM = \frac{1}{2}BC = AN$, and $ZN = \frac{1}{2}GH = AM$; therefore AMZN is a parallelogram. Hence MN bisects AZ. But VW joins the middle points of two sides of the triangle AFI; therefore VW bisects, and is itself bisected by, the median AZ.

§ 36. If O_1 is the middle point of AZ, then O_1M is parallel to AU and equal to half of it.

For in the triangle ZAU, O_1 is the middle point of ZA and M is the middle point of ZU.

Similarly O_1N is parallel to AU' and equal to half of it. Hence $UU' = 2MN$.

§ 37. The middle point of any side of a triangle is equidistant from the centres of the squares described in the same manner on the other two sides.

For O_1 is the middle point of VW and O_1M, which is parallel to AU, is perpendicular to VW; therefore $MV = MW$.

§ 38. If V', W' be the projections of V, W on BC, then $VV' = XV'$ and $WW' = XW'$.

For the quadrilateral AXCV is encyclic, since the angles AXC, AVC are each right; therefore the angle VXC = the angle VAC = half a right angle; therefore $VV' = XV'$. Similarly $WW' = XW'$.

§ 39. The centroid of the triangle UVW is the same as the centroid of the triangle ABC, and that of U'VW the same as that of AHG.

The distance of the centroid of the triangle UVW from BC is $\frac{1}{3}(VV' + WW' - UM)$, and the distance of the centroid of the

Digitized by Google

triangle ABC from BC is $\frac{1}{3}$ AX. It remains, therefore, to prove that $VV' + WW' - UM = AX$.

From the encyclic quadrilateral AXCV, by application of Ptolemy's theorem, there results

$$AX \cdot CV + CX \cdot AV = VX \cdot AC.$$

Now	$CV =$	$AV = AC/\sqrt{2}$;
therefore	$AX + CX$	$= VX\sqrt{2} = 2VV'$.
Similarly	$AX + BX$	$= 2WW'$.
But	BC	$= 2UM$;
therefore	$2(VV' + WW' - UM)$	$= 2AX + BX + CX - BC$
		$= 2AX$.

Hence also the distance of the centroid of UVW from either AB or CA is the same as the distance of the centroid of ABC from AB or CA. The two triangles consequently have the same centroid.

§ 40. O_1 is the centre of a circle which passes through the following ten points :—V, W, M, X, N, Y, the feet of the perpendiculars from V on WU, WU', and from W on VU, VU'.

The circle with O_1 as centre and OV or OW as radius is readily seen to pass through the feet of the four perpendiculars from V and W. Also this circle will pass through N and Y, if it can be shown to pass through M and X.

Now O_1M is half of AU, and $AU = VW$; therefore O_1M is half of VW ; therefore the circle passes through M.

But since AX and ZM are perpendicular to BC, and $O_1A = O_1Z$, therefore $O_1X = O_1M$; and therefore the circle passes through X.

[The circle on VW as diameter can be proved to pass through X thus : the angle VXC is half a right angle, by § 38, and so is the angle WXB ; therefore the angle VXW is a right angle.]

If O_2, O_3 be the middle points of WU, UV then, O_2, O_3 will be the centres of two other ten point circles.

The Potential of a Spherical Magnetic Shell deduced from the Potential of a Coincident Layer of Attracting Matter.

By A. C. Elliott, B.Sc., C.E.

This is the problem of § 670 in Clerk Maxwell's *Electricity and Magnetism*. The author proposes to proceed by another method and to obtain the result in a different form. Let O be the centre of the spherical surface on which the shell lies and Z the point where the

Digitized by Google

magnetic potential V_m is to be found. Also let ϕ be the strength of the shell (magnetic moment per unit area), a its internal, and $a+\delta a$ its external radius. To represent the magnetic distribution let a layer of negative magnetic matter of density σ cover the inside face, and a corresponding positive layer the outside face. Finally, let Z be without the matter of the shell and on the positive side.

Since in a magnet the total quantity of magnetic matter is zero, these hypothetical layers are subject to the condition

$$a^2\sigma = \text{const.} \qquad (1).$$

Let V be the potential at Z due to a single layer of density σ and radius a. The magnetic potential V_m is the sum of the potentials due to the two imaginary layers; and hence by Taylor's theorem

$$V_m = V + \frac{dV}{da}\delta a + \frac{dV}{d\sigma}\delta\sigma - V$$

$$= \frac{dV}{da}\delta a + \frac{dV}{d\sigma}\delta\sigma \qquad (2).$$

From the nature of the potential function

$$V = A\sigma \qquad (3),$$

where A is independent of σ—in fact, the potential for unit density.

From (1) $$\delta\sigma = -\frac{2\sigma}{a}\delta a$$

From (3) $$\frac{dV}{d\sigma} = A = \frac{V}{\sigma}$$

Therefore (2) becomes

$$V_m = \frac{dV}{da}\delta a - \frac{2V}{a}\delta a$$

or since δa is an independent variation

$$V_m = a^2\frac{d}{da}\left(\frac{V\delta a}{a^2}\right) \qquad (4).$$

But $$V\delta a = A\sigma\delta a = A\phi.$$

Hence if P be the potential at Z due to a layer of density numerically equal to ϕ

$$V_m = a^2\frac{d}{da}\left(\frac{P}{a^2}\right) \qquad (5).$$

Calling r the distance OZ, Maxwell obtains

$$V_m = -\frac{1}{a}\frac{d}{dr}(Pr) \qquad (6).$$

It appears therefore that the operations denoted by (5) and (6) respectively are equivalent. The first might sometimes be the more convenient to use—for instance, Maxwell, § 695, eqn. 6'.

Digitized by Google

Second Meeting, December 9th, 1887.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

A Method of Transformation in Geometry.

By R. E. ALLARDICE, M.A.

In spherical, as in plane geometry, transformation by inversion and by similar figures may be used for the deduction of new theorems from known ones, the two methods, however, in the former case becoming identical; and like all other propositions and methods in spherical geometry, these methods of transformation may be dualized. The transformation indicated as the dual of these in the case of the spherical surface is also applicable *in plano*. The sequel is an account of this method and of some results that may be obtained by means of it.

Let XY (fig. 5) be a fixed straight line, P the curve to be transformed, AB a tangent to P meeting XY in B and making with XY an angle θ, BC a straight line making with XY an angle ϕ; then if θ and ϕ are connected by the relation $\tan\frac{1}{2}\phi = k\tan\frac{1}{2}\theta$, the envelope of BC is the transformed curve. This transformed curve may be called, for reference, the second inverse of P. [The phrase "second inverse" of P would naturally mean the inverse of the inverse of P; but, as this is the curve P itself, the phrase is not much used in this sense, and may therefore be employed to designate the curve obtained from P by the method of tranformation that is dual to the method of inversion.]

The second inverse of a circle is another circle such that the axis of transformation is the radical axis of the two circles.

This theorem is most easily established by proving first the converse, which is as follows:—

Digitized by Google

Let P (fig. 6) be a point in the radical axis of two circles; PQ and PR tangents to these circles; to prove that the ratio $\tan\frac{1}{2}QPE : \tan\frac{1}{2}RPE$ is constant, independent, that is, of the position of P.

Let PC and PD, the bisectors of the angles at P, meet the central axis of the circles in C and D respectively; then the theorem is equivalent to this, that the ratio CE : ED is constant.

Let QA and PC meet in O.

Since $\angle CPQ = \angle CPE$, $\therefore AC = AO$.

Now $CE : EP = QO : QP$,

$\therefore CE : CE + QA + AC = EP : EP + QP$,

$\therefore CE : EH = EP : EP + QP$;

similarly $DE : EK = EP : EP + RP$

$= EP : EP + QP$.

$\therefore CE : DE = EH : EK$

$=$ constant.

The converse of this theorem is obviously true; that is, if the circle A and the line PE be given and if PR be always drawn so that $\tan\frac{1}{2}RPE = k\tan\frac{1}{2}QPE$, where k is a constant, then PR will envelope a circle, PE will be the radical axis of this circle and the given circle, and the ratio EK : EH will be equal to k.

It may be noticed here that although the transformation was originally defined analytically, a purely geometrical form may be given to it; for (fig. 6) the conditions

$\tan\frac{1}{2}RPE = k\tan\frac{1}{2}QPE$ and $CE = kED$ are identical.

Expression for the radius of the transformed circle.

In fig. 6 let $AE = d$, $AH = r$, $BE = d'$, $BK = r'$; then by the property of the radical axis

$$EK.EK' = EH.EH',$$

but $EK = kEH$,

$\therefore EK' = EH'/k$.

Hence $d' + r' = k(d + r)$,

$d' - r' = (d - r)/k$;

$\therefore 2r' = r(k + 1/k) + d(k - 1/k)$.

Digitized by Google

Hence the radius of the transformed circle will be zero, or this circle will reduce to a point, if

$$k^2 = (d-r)/(d+r),$$

which gives real values for k if the given circle does not cut the axis of transformation.

Another demonstration may be obtained by using a special kind of co-ordinates. Thus (fig. 7) consider a circle of radius a, the distance of the centre of which from a fixed axis is d. Let the tangent at any point P make an angle θ with the axis, and let r be the length of the part of this tangent intercepted by the axis; then

$$r\sin\theta = d + a\cos\theta,$$
$$\therefore \quad r = d\operatorname{cosec}\theta + a\cot\theta,$$

is the equation to the circle in this system of co-ordinates.

This equation may be written

$$r = d\left(1+\tan^2\frac{1}{2}\theta\right)/2\tan\frac{1}{2}\theta + a\left(1-\tan^2\frac{1}{2}\theta\right)/2\tan\frac{1}{2}\theta$$

or,
$$r = \left\{(d+a)+(d-a)\tan^2\frac{1}{2}\theta\right\}/2\tan\frac{1}{2}\theta.$$

The equation is transformed by writing

$$k\tan\frac{1}{2}\phi \text{ in place of } \tan\frac{1}{2}\theta.$$

This gives
$$r = \left\{(d+a)+(d-a)k^2\tan^2\frac{1}{2}\phi\right\}/2k\tan\frac{1}{2}\phi$$

or
$$r = \left\{(d+a)/k+(d-a)k\tan^2\frac{1}{2}\phi\right\}/2\tan\frac{1}{2}\phi,$$

which represents a circle. If δ be the distance of its centre from the axis and α its radius, then

$$\delta + \alpha = (d+a)/k,$$
$$\delta - \alpha = (d-a)k,$$

results which agree with those obtained by the former method.

The principal characteristic of this method of transformation is that the length of the tangent from a point in the axis remains unaltered. This is obvious in the case of the circle from the demonstration given above; and, in the case of any other curve, may be seen from the fact that, if a circle be drawn to touch the curve at its point of contact with a given tangent, and the whole figure be transformed, the transformed circle will touch the other transformed

Digitized by Google

curve at its point of contact with the line into which the given tangent transforms. A direct proof of this property by means of infinitesimals may also be given, as follows:—

Let AP, BP, (fig. 8) be two consecutive tangents; AQ, BQ, the transformed tangents; let $PAX = \theta$, $QAX = \phi$; draw AN and AM perpendicular to BP and BQ respectively; then

$$AP = AN/APB = AN/d\theta = AB\sin\theta/d\theta.$$

Now, $\tan\frac{1}{2}\phi = k\tan\frac{1}{2}\theta$, $\therefore \sec^2\frac{1}{2}\phi d\phi = k\sec^2\frac{1}{2}\theta d\theta$;

$$\therefore AP = AB\sin\theta/d\theta$$
$$= 2AB\tan\frac{1}{2}\theta/\sec^2\frac{1}{2}\theta d\theta$$
$$= 2AB\tan\frac{1}{2}\phi/\sec\frac{1}{2}\phi d\phi$$
$$= AQ.$$

If a tangent to a given curve P makes an angle θ with a given axis, it may also be considered as making an angle $\pi + \theta$ with that axis. Hence if the angle of inclination ϕ of the transformed tangent is determined by the relation $\tan\frac{1}{2}\phi = k\tan\frac{1}{2}\theta$, there are two possible values of ϕ, and therefore two possible positions of the transformed tangent. Each of these tangents has an envelope, and hence a curve may, in general, be transformed in two different ways for the same value of k. In the case of the circle, these two transformations may easily be discussed geometrically; the following is an analytical investigation.

Let the co-ordinates of any point on the circle (figs. 9 and 10) be expressed in terms of the angle θ contained by a radius to the point and a radius parallel to the axis of transformation, namely

$$x = a + r\cos\theta,\ y = b + r\sin\theta.$$

Let ϕ be the angle made by a tangent to the circle with the axis of transformation, and let $\tan\frac{1}{2}\psi = k\tan\frac{1}{2}\phi$.

In fig. 9, $\phi = \theta + \frac{\pi}{2}$; in fig. 10, $\phi = \theta + \frac{3\pi}{2}$. Consider fig. 9.

$$x = a + r\cos\theta = a + r\sin\phi$$
$$y = b + r\sin\theta = b - r\cos\phi$$

The equation to the tangent at (x_1, y_1) is

$$y - y_1 = (x - x_1)\tan\phi.$$

Digitized by Google

When $y=0$ $\qquad x'=x_1-y_1\cot\phi$;
and the transformed curve is the envelope of

$$y=(x-x')\tan\psi, \text{ where } \tan\frac{1}{2}\psi=k\tan\frac{1}{2}\phi;$$

that is
$$y=(x-x_1+y_1\cot\phi)\tan\psi$$
$$=(x-a+b\cot\phi-r\operatorname{cosec}\phi)\tan\psi.$$

If $\tan\frac{1}{2}\phi=\lambda$, this equation reduces to

$$\{k^2y-(r+b)k\}\lambda^2+2(x-a)k\lambda+(b-r)k-y=0;$$

the envelope of which is given by

$$(x-a)^2k^2-\{k^2y-(r+b)k\}\{(b-r)k-y\}=0$$

that is,

$$(x-a)^2+\left\{y-\frac{1}{2}b(k+1/k)+\frac{1}{2}r(k-1/k)\right\}^2=\frac{1}{4}\{b(k-1/k)-r(k+1/k)\}^2.$$

To get the equation for the case of fig. 10, the sign of r must be changed throughout.

Let δ and ρ be the distance of the centre from the axis and the radius for fig. 9, and δ' and ρ' the corresponding quantities for fig. 10. Then

$$\delta=\frac{1}{2}b(k+1/k)-\frac{1}{2}r(k-1/k);\quad \delta'=\frac{1}{2}b(k+1/k)+\frac{1}{2}r(k-1/k);$$

$$\rho=\frac{1}{2}b(k-1/k)-\frac{1}{2}r(k+1/k);\quad \rho'=\frac{1}{2}b(k-1/k)+\frac{1}{2}r(k+1/k).$$

$$\delta+\rho=(b-r)\,k;\qquad \delta'+\rho'=(b+r)k;$$
$$\delta-\rho=(b+r)/k;\qquad \delta'-\rho'=(b-r)/k.$$

Hence, for any given value of k, a circle A may be transformed into two distinct circles B and C; the difference being that for one of the circles, B say, the tangent which envelopes A is supposed to move round A in the positive direction, while for C the enveloping tangent moves round A in the negative direction. Only one of the circles B and C may be considered at a time, or both at once, as is convenient. The two forms of transformation may be distinguished as direct and inverse.

In fig. 11, A and B are two circles whose direct common tangents intersect in the axis of transformation. A transforms into the two circles C and C', and B into D and D'. As is shown in the figure, a direct common tangent to A and B transforms into a direct common tangent to C and D, and also into a direct common tangent to C' and D'; while an inverse common tangent to A and B transforms

Digitized by Google

into a common tangent to C and D′ and also into a common tangent to C′ and D; that is to say, if two circles are transformed either both directly or both inversely, the *direct* common tangents transform into common tangents, while if one of the circles is transformed directly and the other inversely, the *inverse* common tangents transform into common tangents. It is obvious that when two circles are transformed the length of the common tangent remains unaltered. This corresponds to the fact that, in transformation by similar figures, the angle of intersection of two curves remains unaltered.

It was shown that, by selection of a proper value of k, any circle may be transformed into a point. In this case any two tangents from a point in the axis of transformation transform into the same straight line; and hence any two circles, the common tangents to which intersect in the axis of transformation may be transformed into points by the same transformation. If an axis of similitude is taken for axis of transformation, any three circles may be transformed into points; and in general any number of circles which possess a common axis of similitude may be so transformed, two circles that lie on the same side of the axis being transformed either both directly or both inversely, and two that lie on opposite sides being transformed, one directly and one inversely. It may be noted that a point transforms into only one circle.

If a single circle is transformed in all possible ways, that is, for all values of k, an infinite series of circles is obtained, consisting of all the circles that cut another system orthogonally. If this system is transformed for any one value of k, it transforms into itself.

If all the points on the circumference of a circle be transformed, the centres of the circles into which they transform will lie on an ellipse.

Let P (fig. 12) transform into a circle of which the centre is P′, and which passes through Q and R,

then
$$QM = PM/k \text{ and } RM = k\,PM$$
$$\therefore\ QM + RM = (k + 1/k)PM$$
$$\therefore\ P'M = \frac{1}{2}(k + 1/k)PM.$$

Hence the locus of P′ is an ellipse, one of the axes of which is equal to the diameter of the circle on which P lies. The other axis is $\frac{1}{2}(k + 1/k)$ times this diameter.

Digitized by Google

EXAMPLES OF THE APPLICATION OF THIS METHOD.

(1) By means of this method of transformation may be found very easily the condition that four straight lines touch the same circle, in terms of the angles that the lines make with the third diagonal of the quadrilateral formed by them.

Let AB (fig. 13) be this third diagonal; and let the circle touched by the four lines be transformed into the point P.

Let PA and PB make angles α and β with AB.

Let the two tangents through A make angles ϕ_1 and ϕ_2, and the two tangents through B angles ψ_1 and ψ_2, with AB.

Then $\tan\frac{1}{2}\alpha = k\tan\frac{1}{2}\phi_1$ $\tan\frac{1}{2}\beta = k\tan\frac{1}{2}\psi_1$

$\tan\frac{1}{2}(\alpha+\pi) = k\tan\frac{1}{2}\phi_2$ $\tan\frac{1}{2}(\beta+\pi) = k\tan\frac{1}{2}\psi_2$.

The required condition is obtained at once, by elimination of α, β and k from these four equations; namely,

$$*\tan\frac{1}{2}\phi_1 \tan\frac{1}{2}\phi_2 = \tan\frac{1}{2}\psi_1 \tan\frac{1}{2}\psi_2.$$

(2) If four circles have a common axis of similitude, they may be transformed into four points. Hence their six common tangents satisfy the relation which connects the six straight joining four points.

It should be noted here, and the remark applies to all cases in which an axis of similitude and a common tangent are referred to, that when the axis of similitude passes through the external centre of similitude of two circles, the direct common tangent is to be understood, and when through the internal centre of similitude the inverse common tangent.

(3) If four circles have a common axis of similitude, and touch a common circle, their common tangents satisfy the relation connecting the diagonals and sides of a quadrilateral inscribed in a circle, namely,

$h^2 = (ac+bd)\ (ad+bc)/(ab+cd)$, $k^2 = (ac+bd)\ (ab+cd)/(ad+bc)$.

(4) Given three fixed circles A, B, C, the locus of the centre of a circle P such that the common tangents to P and A, P and B, P

---

* This result is implicitly contained in a paper by me in vol. iii. of the *Proceedings*, page 60.

Digitized by Google

and C, are equal is a straight line perpendicular to the axis of similitude of the circles A, B, C.

This result is obtained by transforming the circle A, B, C, into three points, and noticing that the system of circles P transforms into a system of concentric circles.

There are four different cases, one arising from consideration of each of the axes of similitude.

The common tangent to either of the circles A, B, C, and the circle P is a maximum when P, A, B, C have a common axis of similitude.

(5) The eight circles touching the circles A, B, C, of the last example are particular cases of the circle P. Hence the centres of these eight circles lie in pairs on straight lines, passing through the radical centre and perpendicular to the axes of similitude of the circles A, B, C.

The two circles that have their centres in the same line perpendicular to one of the axes of similitude have obviously that axis for their radical axis. (See *Casey's Sequel to Euclid*, Third Edition, page 121.)

(6) The last example gives a method for constructing the circles touching three given circles.

(7) Let three points be transformed with every possible value of k. A one-fold infinity of systems of three circles will be obtained, possessing the properties that one of the axes of similitude of all the systems and the centres of similitude lying in that axis are fixed, and that the lengths of the tangents to the circles from the fixed centres of similitude are invariable. The radical centre and the centres of the common tangent circles for all the systems will lie in a straight line perpendicular to the fixed axis of similitude. [To each axis of similitude correspond two tangent circles.]

(8) The locus of the centres of all the circles that touch a given circle and have a common axis of similitude is an ellipse.

Digitized by Google

Third Meeting, January 13th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

On Stirling's approximation to $n!$ when n is large.

By R. E. ALLARDICE, M.A.

The approximation is

$$n! = (n/e)^n \sqrt{2\pi n}.$$

We have

$$\left(\frac{n}{e}\right)^n = \left(\frac{n-1}{e}\right)^{n-1} \cdot \frac{1}{e} \cdot \left(\frac{n}{n-1}\right)^n \cdot \left[n(n-1)\right]^{\frac{1}{2}} \cdot \left(\frac{n-1}{n}\right)^{\frac{1}{2}}$$
$$= \left(\frac{n-1}{e}\right)^{n-1} \cdot \frac{1}{e} \cdot \left(\frac{n}{n-1}\right)^{n-\frac{1}{2}} \cdot \left[n(n-1)\right]^{\frac{1}{2}}.$$

Now assume

$$\left(\frac{n}{n-1}\right)^{n-\frac{1}{2}} = e^x$$

$$\therefore\ x = (n-\tfrac{1}{2}) \log \frac{n}{n-1}$$
$$= (-n+\tfrac{1}{2}) \log\left(1 - \frac{1}{n}\right)$$
$$= (-n+\tfrac{1}{2})\left(-\frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2} - \frac{1}{3}\cdot\frac{1}{n^3} - \ \ldots\ \ldots\right)$$
$$= 1 + \frac{1}{12}\cdot\frac{1}{n^2} + \frac{B}{n^3} + \ \ldots\ \ldots$$
$$= 1 + \frac{A}{n^2}$$

where A differs from 1/12 by a quantity of the order of $1/n$, since the above series for x is convergent; and since n is large, A may be taken to be 1/12.

Hence, $$\left(\frac{n}{e}\right)^n = \left(\frac{n-1}{e}\right)^{n-1} \cdot \frac{1}{e} \cdot e^{1 + A/n^2} \cdot \left[n(n-1)\right]^{\frac{1}{2}}$$
$$= \left(\frac{n-1}{e}\right)^{n-1} \cdot e^{A/n^2} \cdot \left[n(n-1)\right]^{\frac{1}{2}}.$$

Digitized by Google

Substituting for n in succession the values, $2n$, $2n-1$ ... ... $n+1$, we get

$$\left(\frac{2n}{e}\right)^{2n} = \left[2n(2n-1) \ \ldots \ \ldots \ (n+1)(2n-1) \ \ldots \ \ldots \ n\right]^{\frac{1}{2}} \cdot \left(\frac{n}{e}\right)^{n} \cdot e^{k}$$

where $$k = \frac{A}{(2n)^2} + \frac{A}{(2n-1)^2} + \ \ldots \ \ldots \ + \frac{A}{(n+1)^2}$$

$$\therefore \quad \sqrt{2}\left(\frac{2n}{e}\right)^{2n} = \left(\frac{n}{e}\right)^{n} \cdot \frac{(2n)!}{n!} \cdot e^{k}.$$

$$\therefore \quad \sqrt{2}\left(\frac{n}{e}\right)^{n} \cdot 2^{2n} = \frac{(2n)!}{n!} \cdot e^{k}$$

$$\therefore \quad n! = \left(\frac{n}{e}\right)^{n} \cdot \sqrt{2} \cdot 2^{2n} \cdot \frac{(n!)^2}{(2n)!} \cdot e^{-k}.$$

Now, Wallis's formula for π is

$$\frac{\pi}{2} = \left[\frac{2.4.6 \ \ldots \ \ldots \ 2n}{1.3.5 \ \ldots \ \ldots \ 2n-1}\right]^2 \cdot \frac{1}{2n+1}$$

or $$\frac{\pi}{2} = \left[\frac{2.4.6 \ \ldots \ \ldots \ 2n}{1.3.5 \ \ldots \ \ldots \ 2n-1}\right]^2 \cdot \frac{1}{2n} \cdot \left(1 + \frac{1}{2n}\right)^{-1}$$

$$\therefore \quad \sqrt{\pi} = \frac{2.4.6 \ \ldots \ \ldots \ 2n}{1.3.5 \ \ldots \ \ldots \ 2n-1} \cdot \frac{1}{\sqrt{n}} \cdot \left(1 + \frac{1}{2n}\right)^{-\frac{1}{2}}$$

Hence we may write

$$\sqrt{\pi} = \frac{2^{2n}(n!)^2}{(2n)!} \cdot \frac{1}{\sqrt{n}} \cdot \left(1 - \frac{1}{4n}\right).$$

This result reduces the above equation for $n!$ to

$$n! = \left(\frac{n}{e}\right)^{n} \cdot \sqrt{2} \cdot \sqrt{\pi n} \cdot e^{-k} \cdot \left(1 + \frac{1}{4n}\right)$$

Now, k lies between $nA/(2n)^2$ and $nA/(n+1)^2$. Hence if we write $n! = \left(\frac{n}{e}\right)^{n} \sqrt{2\pi n}$, the ratio of the error to the value found for $n!$ is of the order of $1/n$.

It may now be shown that in the above approximation to π, the error made bears to π a ratio of the order of $1/n$.

In the value assumed for π

$$\frac{\pi}{2} = \left[\frac{2.4.6 \ \ldots \ \ldots \ 2n}{1.3.5 \ \ldots \ \ldots \ 2n-1}\right]^2 \cdot \frac{1}{2n+1}$$

the factor u has been neglected, where

$$u = \frac{(2n+2)^2}{(2n+1)(2n+3)} \cdot \frac{(2n+4)^2}{(2n+3)(2n+5)} \ \ldots \ \ldots .$$

Now, $$\frac{(2n+2)^2}{(2n+1)(2n+3)} = \frac{4n^2+8n+4}{4n^2+8n+3} = 1 + \frac{1}{(2n+1)(2n+3)}$$

Digitized by Google

$$\therefore \quad u = \left[1 + \frac{1}{(2n+1)(2n+3)}\right]\left[1 + \frac{1}{(2n+3)(2n+5)}\right] \ \ldots \ \ldots .$$

$$\therefore \quad \log u = \frac{1}{(2n+1)(2n+3)} + \frac{1}{(2n+3)(2n+5)} + \ \ldots \ \ldots$$

$$- \tfrac{1}{2}\cdot\left[\frac{1}{(2n+1)(2n+3)}\right]^2 - \tfrac{1}{2}\left[\frac{1}{(2n+3)(2n+5)}\right]^2 - .$$

Now $$\frac{1}{(2n+1)(2n+3)} = \tfrac{1}{2}\left[\frac{1}{2n+1} - \frac{1}{2n+3}\right].$$

Hence, neglecting small quantities of a higher order than $1/n$, we may write

$$\log u = 1/2(2n+1)$$

$$\therefore \quad u = e^{1/2(2n+1)}$$

$$= 1 + 1/2(2n+1) + \ \ldots \ \ldots ;$$

which shows that the error made in the above approximation to π is of the order stated.

[Another elementary proof of Stirling's theorem, by Mr J. W. L. Glaisher, is given in the *Quarterly Journal of Mathematics*, vol. xv., p. 57.]

A Device for the Analysis of Intervals and Chords in Music.

By A. Y. Fraser, M.A., F.R.S.E.

§ 1. The device here described has been found to simplify greatly the "somewhat laborious discussion" of the different musical intervals as given, say, in Sedley Taylor's *Sound and Music* (chap. viii.) or in Helmholtz's *Sensations of Tone*. It has been found particularly helpful in giving an account, necessarily rapid, of the nature of harmony to classes studying sound as a part of physics, from whom much familiarity with musical terms and notation is not to be expected.

§ 2. The leading idea of the device is that the octave being what may be called a periodic phenomenon, should be represented on a circular or spiral curve, and not, as is usual, on a straight vertical line. Such a representation of four octaves is given in figure 14. (To read this and the other figures, begin at the extreme right and

Digitized by Google

follow the spiral round counter-clock-wise.) The compactness of this representation as compared with that on a straight line is at once obvious. The different stretches of intervals are represented by the different sizes of angles (which are, as shown, proportional to the numbers 45, 40, and 24). The various observations that have to be made regarding intervals, for example, that the fourth is the inversion of the fifth, that the interval D to *m* is equal to the interval *s* to *t*, and so on, can be got from this figure, at least as well as from the usual vertical scale. It remains to show how this diagram can be developed to exhibit readily the nature of harmonies, that is, of two or more notes sounded together; but before this can be shown, some further preliminary explanation is necessary.

§ 3. When two *simple* tones are sounded together, the effect is agreeable* or the reverse, according to the interval between them. The effect is found to be unpleasant if the interval be less than a minor third; the interval of a tone being less disagreeable than the interval of a semi-tone, and any disagreeable interval becoming less disagreeable if both notes are taken at a higher pitch. This unpleasantness has been shown to be due to rapid waxings and wanings of sound, called *beats*, as the vibrations of the two notes assist or oppose each other. The interval of a minor third is regarded, in fact, as the smallest smooth interval. Such an interval can be readily distinguished on our diagram, for it is represented by what is almost exactly a right angle.

§ 4. So much for *simple* tones; but, if we except the tuning fork, which, in certain circumstances does so, no musical instrument gives out simple tones; every tone is *compound*—is, in fact, a congeries of simple tones. Such a compound tone is made up as follows: There is (A) a loud sounding simple tone and (B) a collection of simple tones higher in pitch than (A), differing in relative intensity, and all in general much weaker than the fundamental tone (A). These

---

* I use the words agreeable and disagreeable to save circumlocution, knowing all the time that musicians object *in toto* to the use of the words. A "disagreeable" interval may be used in a musical composition like a "disagreeable" incident in any other work of art, with the most happy general effect. Absolute smoothness in music, like absolutely pure water, is neither quite attainable, nor at all pleasing to the taste if attained.

Digitized by Google

component simple tones are called *partial tones*, the fundamental tone (A) being called the *prime partial*, and the rest (B) the *upper partials*. If the frequency of vibration of the prime partial be 1, then the series of upper partials have frequencies 2, 3, 4, 5, 6, &c., and thus if C represent the pitch of the prime partial tone, the whole series is (Fig. 15)

1,	2,	3,	4,	5,	6, &c.
C	C′	G′	C″	E″	G″, &c.

As a rule, beyond the last note given above, the upper partials are so weak as to be practically non-existent. The *quality* of any tone is determined by the number and relative intensities of the partial tones composing it, while the *pitch* of a tone is that of its prime partial.

§ 5. The last two sections are, of course, an epitome of facts, due for the most part to Helmholtz, and quite well known, but necessary for the understanding of the contrivance now to be described. In our examples we shall suppose the tones to be each made up of the six partials given above. They are the most important*, and the analysis of an interval with any other combination of partial tones is in principle quite the same.

§ 6. Figure 15 represents a diagram drawn on a large sheet of cardboard. Behind the cardboard is glued a strip of wood, which affords a support to a pin (say two inches of an ordinary pen holder), which projects from the centre of the spiral at right angles to the surface of the cardboard. This diagram, in which the six arrowheads represent six partials, will be taken to stand for all the partial tones of the lower of two notes whose combined effect is to be analysed.

Figure 16 represents a moveable arrangement to be used in connexion with figure 15. In the middle is a circle of wood with a hole in its centre, so that it can be made to rotate stiffly about the pin in fig. 15. This circle of wood carries radially three rods (steel knitting wires), and on each rod there slides a strip of cardboard cut out to represent one, two, and three arrowheads. The size of the arrangement is such that the whole can be made to cover exactly the corres-

---

* "The sounds of most musical instruments practically contain only the first six partial-tones."—Sedley Taylor, *Sound and Music*, 2nd edit., p. 167.

Digitized by Google

ponding parts of fig. 15. Figure 16 is to be employed to represent the higher of two notes that are sounded together.

§ 7. Suppose, now, the framework of fig. 16 to be superposed on fig. 15 so as to coincide with it exactly. This will represent *unison*, that is, the sounding together of two notes of the same pitch and quality, with, therefore, complete coincidence in all the partial tones. Turn now the upper part a degree or two in either direction, and this will, of course, put every arrowhead of the one a degree or two from the corresponding arrowhead of the other. The interpretation of the diagram now is that there are six beating pairs of partials, which shows the *definiteness* of the unison combination—a chord being definite when it is bounded on each side by sharp discords.

§ 8. Again, let us look at the nature of the *major third* interval. Turn* round the moving part of our diagram till its "C" coincide with the "E" of the diagram on fig. 15. The dotted lines on fig. 17 show the position taken up by the moving part.

An examination of the diagram (fig. 17) shows that there is beating between the 4th partial of C and the 3rd of E, which two are a semi-tone apart, and also between the 6th of C and the 5th of E, also a semi-tone apart, but at a much higher pitch, and on that account, as well as from the fact that they will in general be weaker, they are a less rough pair. So much for the beating, due to partials in the major third.

If, now, the upper of the two notes be raised an octave (or the lower depressed an octave)—that is, if a male and a female voice sing the notes—the above beating intervals are cut out, and thus, so far as the first six partials are concerned, the *major tenth* (for such the interval now is) is a smooth interval. This bears out the known fact that a "major third" is smoother when taken by a male and a female voice than when taken by two females.

§ 9. We need not go into further detail in this direction. By

---

* As the moving part turns round to the left, its arrowheads project beyond the lines of the spiral to which they belong, and have either to be pushed back by hand or to be pulled back by strings passing round the fixed pin in the centre and shortening by being wound up. A small piece of india-rubber cord joining the point of each of the three cardboard strips of the outer extremity of its rod, will keep the string tight.

Digitized by Google

turning round the movable part (fig. 16, that is to say) every combination of two notes can be exhibited; angles less than a right angle—that is, intervals less than a minor third—at once catch the eye, and the numbers written on the arrowheads tell without trouble what partials are concerned. The roughness of any interval being thus ascertained, we can examine into the definiteness (due really to coinciding partials) by turning the upper part a degree or two out of position.

§ 10. One or two miscellaneous suggestions may be useful.

By using two moving parts like fig. 16 in conjunction with fig. 15, combinations of three notes can be studied. This requires no further illustration here.

A diagram such as fig. 14, with a spiral making four or five circuits, can be used as a repository of a good deal of information. One circuit may have the notes named as D, *r*, *m*, &c., while the next bears the letters C, D, E, &c. A third circle might carry the vibration ratios 1, $\frac{9}{8}$, $\frac{5}{4}$, &c., while the fourth might give a standard series of vibration numbers. The outermost circuit of a fair sized spiral would afford abundance of room for particulars concerning the various sharpened and flattened notes.

Our analysis has left combinational tones out of account, but the insertion of vibration numbers, as has just been suggested—even if a special diagram were devoted to this alone—would make the placing of the combinational tones an easy enough affair.

The method of using the diagram as described above is not the only way of turning the idea to account. A private student may use in connection with fig. 15 semi-transparent ground glass, which can be written on, or common glass with radial paper lines gummed on, or even a copy of fig. 16 cut out in cardboard; while for demonstration to a large audience the diagrams done on glass can be projected in the usual way by the lantern.

Digitized by Google

Fourth Meeting, February 10th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

On the inequality $mx^{m-1}(x-1) \gtrless x^m - 1 \gtrless m(x-1)$ and its consequences.

By PROFESSOR CHRYSTAL.

§ 1. The object of this note is to establish the above inequality in as general a form as possible, and to prove by means of it two of the principal propositions in the theory of inequalities, one of which is usually proved by means of infinite series. The logical advantage in making the theory of inequalities independent of that of infinite series is obvious, when it is remarked that the discussion of the convergency of infinite series is strictly speaking a part of the theory of inequalities.

§ 2. *If x, p, q, are all positive, and p and q are integers, then $(x^p-1)/p \gtrless (x^q-1)/q$ according as $p \gtrless q$.*

Since p and q are integers

$$(x^p-1)/p \gtrless (x^q-1)/q,$$

as $$q(x^p-1) \gtrless p(x^q-1),$$

as

$$(x-1)\{q(x^{p-1}+x^{p-2}+\ldots+x+1)-p(x^{q-1}+x^{q-2}+\ldots+x+1)\} \gtrless 0.$$

Suppose $p>q$; and denote the expression on the left side of the last inequality by X. Then

$$X=(x-1)\{q(x^{p-1}+x^{p-2}+\ldots+x^q)-(p-q)(x^{q-1}+x^{q-2}+\ldots+x+1)\}.$$

Now, if $x>1$, $x^{p-1}+x^{p-2}+\ldots+x^q>(p-q)x^q$,

and $x^{q-1}+x^{q-2}+\ldots+1<qx^{q-1}$;

and therefore $X>(x-1)\{q(p-q)x^q-(p-q)qx^{q-1}\}$

$$>q(p-q)x^{q-1}(x-1)^2$$

$$>0.$$

Again, if $x<1$, $X>(x-1)\{q(p-q)x^{p-1}-(p-q)q\}$

$$>q(p-q)(x-1)(x^{p-1}-1)$$

$$>0.$$

Digitized by Google

Hence, in both cases, $(x^p-1)/p > (x^q-1)/q$.

By the same reasoning, if $q>p$,

$$(x^q-1)/q > (x^p-1)/p\,;$$

that is, if $p<q$, $\quad (x^p-1)/p < (x^q-1)/q$.

§ 3. If x is positive, and different from unity, then

$$mx^{m-1}(x-1) > x^m-1 > m(x-1)$$

unless m lie between 0 and 1, in which case

$$mx^{m-1}(x-1) < x^m-1 < m(x-1).$$

From last paragraph we have

$$(\xi^p-1) \gtrless (p/q)(\xi^q-1) \qquad (1)$$

according as $p \gtrless q$; where ξ is any positive quantity, different from unity, and p and q are positive integers. In (1) we may put $x^{1/p}$ for ξ, where x is any positive quantity different from unity, the real positive value of the q^{th} root being taken; and we may put m for p/q, where m is any positive commensurable quantity.

The inequality then becomes

$$x^m-1 \gtrless m(x-1) \qquad (2)$$

according as $m \gtrless 1$; which is part of the theorem.

In (2) put $1/x$ for x, then

$$(1/x)^m-1 \gtrless m(1/x-1)$$

or $\quad mx^{m-1}(x-1) \gtrless x^m-1. \qquad (3)$

according as $\quad m \gtrless 1.$

The theorem is thus established for positive values of m.

Next, let $m=-n$, then

$$x^{-n}-1 \gtrless (-n)(x-1),$$

according as $\quad 1-x^n \gtrless -nx^n(x-1),$

according as $\quad x^n-1 \gtrless nx^n(x-1),$

according as $\quad nx^{n+1}-nx^n \gtrless x^n-1,$

according as $\quad (n+1)x^n(x-1) \gtrless x^{n+1}-1.$

Now, since n is positive, $n+1>1$; therefore by (3)

$$(n+1)x^n(x-1) > x^{n+1}-1\,;$$

and therefore $\quad x^{-n}-1 > (-n)(x-1). \qquad (4)$

In (4) write $1/x$ for x, then

$$(1/x)^{-n}-1 > (-n)(1/x-1)$$

therefore $\quad x^{-n}-1 < (-n)x^{-n-1}(x-1)$

that is, $\quad (-n)x^{-n-1}(x-1) > x^{-n}-1.$

Hence, if m is negative,

$$mx^{m-1}(x-1) > x^m-1 > m(x-1)\,;$$

which completes the demonstration.

Digitized by Google

§ 4. *The arithmetic mean of n positive quantities is not less than the geometric mean.*

Let us suppose the theorem to hold for n quantities a, b, c, ... k; and let l be one more.

By hypothesis, $(a+b+c+\ \dots\ \dots\ +k)/n \not< (abc\ \dots\ \dots\ k)^{1/n}$

that is, $a+b+c+\ \dots\ \dots\ +k \not< n(abc\ \dots\ \dots\ k)^{1/n}$

Therefore $a+b+c+\ \dots\ \dots\ +k+l \not< n(abc\ \dots\ \dots\ k)^{1/n}+l.$

Now, $n(abc\ \dots\ \dots\ k)^{1/n}+l \not< (n+1)(abc\ \dots\ \dots\ kl)^{1/(n+1)},$

provided $n(abc\ \dots\ \dots\ k/l^n)^{1/n}+1 \not< (n+1)(abc\ \dots\ \dots\ kl/l^{n+1})^{1/(n+1)},$

$\not< (n+1)(abc\ \dots\ \dots\ k/l^n)^{1/(n+1)};$

that is, provided $n\xi^{n+1}+1 \not< (n+1)\xi^n,$

where $\xi^{n(n+1)} = abc\ \dots\ \dots\ k/l^n,$

that is, provided $(n+1)\xi^n(\xi-1) \not< \xi^{n+1}-1,$

which is true, by the theorem of last paragraph.

Hence, if the theorem hold for n quantities, it will hold for $n+1$; and it is obviously true for two quantities, and hence it is true generally.

Corollary.—If a, b, ... k are n positive quantities, and p, q, ... t, n positive commensurable quantities, then

$$\frac{pa+qb+\ \dots\ \dots\ +tk}{p+q+\ \dots\ \dots\ +t} > (a^p b^q\ \dots\ \dots\ k^t)^{1/(p+q+\ \dots\ \dots\ +t)}$$

It is obvious that we are only concerned with the ratios of the quantities p, q, ... t; and we may, therefore, suppose these quantities to be integral. The theorem is thus seen to be a particular case of that just proved—namely, that the arithmetic mean of $p+q+\ \dots\ \dots\ t$ positive quantities, of which p are equal to a, q to b, and so on, is greater than their geometric mean.

§ 5. *If a, b, ... k, are n positive quantities, and p, q, ... t, are n commensurable quantities, then*

$$\frac{pa^m+qb^m+\ \dots\ \dots\ +tk^m}{p+q+\ \dots\ \dots\ +t} \lessgtr \left(\frac{pa+qb+\ \dots\ \dots\ +tk}{p+q+\ \dots\ \dots\ +t}\right)^m \qquad (1)$$

according as m does not or does lie between 0 and 1.

If we denote $p(p+q+\ \dots\ \dots\ +t)$, $q(p+q+\ \dots\ \dots\ +t)$, etc., by λ, μ, ... τ; and $a/(\lambda a+\mu b+\ \dots\ \dots\ +\tau k)$, $b/(\lambda a+\mu b+\ \dots\ \dots\ +\tau k)$, etc., by x, y, ... w, then

$$\lambda+\mu+\ \dots\ \dots\ +\tau = 1$$

$$\lambda x+\mu y+\ \dots\ \dots\ +\tau w = 1.$$

Dividing both sides of (1) by $\{(pa+qb+\ \dots\ tk)/(p+q+\ \dots\ +t)\}^m$ we have to prove that

Digitized by Google

$$\lambda x^m + \mu y^m + \ \dots\ \dots\ + \tau w^m \lessgtr 1$$

according as m does not or does lie between 0 and +1.

Now, if m does not lie between 0 and 1,

$$x^m - 1 \lessgtr m(x-1),\ y^m - 1 \lessgtr m(y-1),\ \text{etc.}$$

Therefore
$$\begin{aligned} \Sigma\lambda(x^m - 1) &\lessgtr \Sigma\lambda m(x-1) \\ &\lessgtr m\{\Sigma\lambda x - \lambda\} \\ &\lessgtr m(1-1) \\ &\lessgtr 0\,; \end{aligned}$$

that is,
$$\Sigma\lambda x^m \lessgtr \Sigma\lambda \lessgtr 1.$$

In like we may show that if m does lie between 0 and 1, then

$$\Sigma\lambda x^m \gtrless 1.$$

Corollary.—If we make $p = q = \ \dots\ \dots\ = t$, we have

$$(a^m + b^m + \ \dots\ \dots\ + k^m)/n \lessgtr \{(a + b + \ \dots\ \dots\ + k)/n\}^m,$$

that is to say, *the arithmetical mean of the m^{th} powers of n positive quantities is not less or not greater than the m^{th} power of their arithmetical mean, according as m does not or does lie between* 0 *and* 1.

§ 6. The inequality just discussed is by no means new, nor has its importance been overlooked, as may be seen from the elegant use of it by Schlömilch in the second chapter of his *Algebraische Analysis*. (See also *Zeitschrift für Mathematik*, Bd. III., p. 387 (1858), and Bd. VII., p. 46 (1862); also G. F. Walker, *Messenger of Mathematics*, vol. XII., p. 37). The inequality has not, however, usually been stated in quite so general a form as the one I have given; and, possibly in consequence, its application to the demonstration of the theorem of § 5 seems to have hitherto escaped notice. This theorem is usually proved by a somewhat awkward combination of induction and the use of infinite series.

The history of the theorem is a little obscure. At first I suspected that it was due to Canchy, but it does not appear in his *Analyse Algébrique* (Paris, 1821). The earliest reference to it which I have discovered was given me by Mr A. Y. Fraser, and occurs in *Problèmes et Développemens sur diverses Parties des Mathématiques*, par M. Reynaud et M. Duhamel. It is there deduced from the maximum and minimum values of $x^m + y^m$ subject to the condition $ax + by = c$. I can scarcely believe that this is the earliest occurrence, and I should be glad if any of our members could furnish me with reference to an earlier.

§ 7. The inequality $mx^{m-1}(x-1) \gtrless x^m - 1 \gtrless m(x-1)$ has the merit

Digitized by Google

of binding together a great variety of algebraical theorems which are usually put before the student without any organic connection whatever; and for this reason I have brought it specially under the notice of the younger members of the Mathematical Society. Its power is not surprising when we reflect on its close connection with the theorem $\mathrm{L}_{x=1}(x^m-1)/(x-1)=m$, which is the fundamental proposition in the differentiation of algebraic functions.

Mr W. Peddie exhibited and described a model of the thermodynamic surface which represents the state of water-substance in terms of pressure, volume, and temperature. Various lines, the equations of which are $\frac{dp}{dt}=\text{const.}$, $\frac{dp}{dv}=\text{const.}$, &c., were drawn upon the surface.

Fifth Meeting, March 9th, 1888.

W. J. Macdonald, Esq., M.A., F.R.S.E., President, in the Chair.

Sur un système de cercles tangents à une circonférence et orthogonaux à une autre circonférence.

Par M. Paul Aubert.

On donne deux cercles S et Σ ayant pour centres les points O et ω, pour rayons r et ρ. Le cercle S est supposé intérieur au cercle Σ, et le point ω intérieur au cercle S.

I. *Tous les cercles T tangents extérieurement au cercle S et orthogonaux au cercle Σ sont tangents à un troisième cercle fixe.*

Figures 18, 19.

Soit T un cercle tangent au cercle S et orthogonal à Σ. Prenons la figure inverse par rapport au point I comme pôle, la puissance d'inversion étant la puissance k^2 du point I par rapport au cercle S. Ce cercle reste invariable, et le cercle Σ se transforme en une droite perpendiculaire au diamètre Iω en un point P′ tel que

$$\mathrm{IP}\cdot\mathrm{IP}'=k^2.$$

Le cercle T se transforme en un cercle T′ tangent au cercle S et coupant à angle droit la droite P′D; son centre est donc sur cette droite, et par suite le cercle T′ est aussi tangent à la circonférence S_1

Digitized by Google

symétrique de S par rapport à la droite P'D. Il en résulte que le cercle T lui-même est tangent à la circonférence U qu'on obtiendrait en transformant S_1, le pôle étant au point I, et la puissance d'inversion égale à k^2. La circonférence ainsi obtenue étant fixe quand on considère tous les cercles tels que T, la proposition est démontrée.

Cherchons son centre V et son rayon.*

Désignons par d la distance $O\omega$. On sait que les circonférences U et S_1 sont homothétiques par rapport au pôle I, et que le rapport de similitude est égal au quotient de la puissance d'inversion par la puissance du pôle relative au cercle S_1.

Ainsi $$\frac{IV}{IO_1} = \frac{\epsilon k^2}{IO_1^2 - r^2} \qquad \text{où } \epsilon^2 = 1.$$

En remarquant que

$$IO + IO_1 = 2IP' = k^2/\rho,$$

puis substituant à IO sa valeur $(d+\rho)$,
et à k^2 l'expression $[(d+\rho)^2 - r^2]$,
il vient

$$IO_1 = \frac{\epsilon[(d+\rho)d - r^2]}{\rho}$$

$$IV = \frac{\epsilon\rho}{r^2 - d^2}[d(d+\rho) - r^2].$$

Son rayon est alors $$R = \frac{\rho^2 r}{r^2 - d^2}.$$

On obtient d'ailleurs immédiatement le point V et le rayon R par une construction géométrique. Ayant en effet démontré l'existence du cercle U tangent à la circonférence T, désignons par B le point de contact, et par A le point où la circonférence T touche le cercle S. Si nous joignons BA, les points B et A étant deux points antihomologues des circonférences S et U, considérées comme inversement homothétiques, la droite BA va couper la ligne des centres VO au centre d'homothétie inverse de ces deux circonférences. Je dis que ce point n'est autre que ω.

En effet, désignons le pour un instant par C. Le produit $CA \times CB$ est constant pour tous les cercles tels que T tangents à S et à U ; soit h^2 sa valeur. Tous les cercles T seront donc orthogonaux à un cercle fixe ayant C pour centre et dont le rayon est h. Mais nous savons déjà que ces cercles T sont orthogonaux au cercle Σ.

* La solution est un peu plus simple si l'on prend pour pôle d'inversion le centre ω du cercle Σ, et pour puissance d'inversion le carré de son rayon.

Digitized by Google

Les deux cercles C et Σ coïncident donc, sans quoi le lieu des centres des circonférences T serait l'axe radical des cercles C et Σ, ce qu'il est absurde de supposer puisque tous ces cercles sont tangents aux circonférences S et U. Donc la droite BA passe par le point ω, qui est le centre d'homothétie inverse des deux circonférences S et U.

Cela posé, pour obtenir le point V, on construira un cercle T quelconque par la méthode connue, puis on joindra ωA qu'on prolongera jusqu'en B. La droite BT coupera le diamètre ωI au point V cherché ; le rayon de la circonférence est BV.

II. *On prend une droite quelconque LL' perpendiculaire à la ligne des centres Oω. Soient ωF, ωG les tangentes menées du point ω à l'un des cercles T ; soient F' et G' les points d'intersection de la droite LL' avec les bissectrices des angles FωO et GωO. Les points F' et G' forment une division homographique quand le cercle T varie.*

Posons $F'L = x$, $G'L = y$; il faut montrer que x et y satisfont à une relation de la forme

$$mxy + nx + py + q = 0.$$

La figure nous donne

$$GT = \rho \operatorname{tg} \frac{G\omega F}{2}.$$

Or,
$$\frac{G\omega F}{2} = \frac{G\omega L}{2} - \frac{F\omega L}{2},$$

et
$$\operatorname{tg} \frac{G\omega L}{2} = \frac{y}{\omega L}, \quad \operatorname{tg} \frac{F\omega L}{2} = \frac{x}{\omega L};$$

d'où, en posant $\omega L = l$,

(1)
$$GT = \rho \frac{l(y-x)}{l^2 + xy}.$$

Cherchons une autre expression de GT. On a

(2)
$$\overline{OT}^2 = \overline{\omega O}^2 + \overline{\omega T}^2 - 2\omega O \cdot \omega T \cos O\omega T.$$

D'ailleurs on sait que

$$\overline{OT}^2 = (r + GT)^2, \quad \overline{\omega O}^2 = d^2, \quad \overline{\omega T}^2 = \overline{GT}^2 + \rho^2,$$

et
$$\cos O\omega T = \cos\left(\frac{G\omega L}{2} + \frac{F\omega L}{2}\right),$$
$$= \cos \frac{G\omega L}{2} \cos \frac{F\omega L}{2} - \sin \frac{G\omega L}{2} \sin \frac{F\omega L}{2}.$$

Remplaçons ces cosinus et sinus par leurs expressions au moyen des tangentes des mêmes arcs, il vient

$$\cos O\omega T = \frac{l^2 - xy}{\sqrt{(x^2 + l^2)(y^2 + l^2)}},$$

Digitized by Google

et la formule (2) devient

$$(r+\mathrm{GT})^2 = d^2+\rho^2+\overline{\mathrm{GT}}^2 - 2d\sqrt{\rho^2+\overline{\mathrm{GT}}^2}\,\frac{l^2-xy}{\sqrt{(x^2+l^2)(y^2+l^2)}}$$

c'est à dire

$$r^2+2r\cdot\mathrm{GT} = d^2+\rho^2-2d(l^2-xy)\frac{\sqrt{\overline{\mathrm{GT}}^2+\rho^2}}{\sqrt{(x^2+l^2)(y^2+l^2)}}.$$

Or, en tenant compte de la relation (1), on a

$$\overline{\mathrm{GT}}^2+\rho^2 = \rho^2\frac{l^2(y-x)^2+(l^2+xy)^2}{(l^2+xy)^2},$$

ou

$$\overline{\mathrm{GT}}^2+\rho^2 = \frac{\rho^2}{(l^2+xy)^2}[l^4+(x^2+y^2)l^2+x^2y^2],$$

$$\overline{\mathrm{GT}}^2+\rho^2 = \frac{\rho^2}{(l^2+xy)^2}(l^2+x^2)(l^2+y^2).$$

Par suite on peut écrire

$$r^2+2r\cdot\mathrm{GT} = d^2+\rho^2-2d\rho\frac{l^2-xy}{l^2+xy}.$$

Si nous égalons l'expression (1) de GT avec celle que fournit cette dernière relation, il vient, après quelques simplifications

(3) $$[(\rho+d)^2-r^2]xy+2lr\rho(y-x)+l^2[(\rho-d)^2-r^2]=0.$$

C'est bien une relation d'homographie entre x et y.

On peut, en remarquant qu' x et y n'y figurent que par les rapports x/l et y/l, ce qu'on pouvait prévoir, poser

$$x/l=x_1,\ y/l=x_2,$$

avec $$\mathrm{A}=(\rho+d)^2-r^2,\ \mathrm{B}=r\rho,\ \mathrm{C}=(\rho-d)^2-r^2,$$

et on a la relation

(4) $$\mathrm{A}x_1x_2+2\mathrm{B}(x_2-x_1)+\mathrm{C}=0.$$

III. *Soient T_1, T_2, T_3, ... T_n une série de cercles T orthogonaux au cercle Σ, le premier aux points A_1 et A_2, le second aux points A_2 et A_3, le troisième aux points A_3 et A_4, ... le $n^{ième}$ aux points A_n et A_{n+1}. La condition nécessaire et suffisante pour que le point A_{n+1} coïncide avec le point A_1 est que l'on puisse satisfaire à une relation de la forme*

$$\mathrm{tg}\frac{k\pi}{n}=\frac{1}{2r\rho}\sqrt{(\rho^2+d^2-r^2)^2-4d^2\rho^2}.$$

Désignons d'une manière générale par son ordonnée x_p le point de rencontre de la bissectrice de l'angle $A_p\omega L$ avec la perpendiculaire LL'. Il est clair que si le point A_{n+1} vient coïncider avec le point A_1, le point x_{n+1} coïncidera avec le point x_1 ; et réciproquement.

Digitized by Google

Nous sommes donc ramené à chercher la condition nécessaire et suffisante pour que la valeur de x_{n+1} obtenue en appliquant successivement la formule (4) à toutes les valeurs consécutives des indices jusqu' à l'indice $(n+1)$, soit égale à la valeur primitive x_1. Nous tirons de (4)

$$x_2 = \frac{2Bx_1 - C}{Ax_1 + 2B}.$$

Désignons par a et b les racines de l'équation du second degré en x obtenue en supposant dans la relation (4) $x_1 = x_2 = x$. On a

$$a + b = 0, \quad Aab = C.$$

Substituons à C cette valeur, il vient

$$x_2 = \frac{2Bx_1 - Aab}{Ax_1 + 2B},$$

d'où
$$\frac{x_2 - a}{x_2 - b} = \frac{2Bx_1 - Aab - Aax_1 - 2Ba}{2Bx_1 - Aab - Abx_1 - 2Bb},$$

ou
$$\frac{x_2 - a}{x_2 - b} = \frac{(2B - Aa)(x_1 - a) - Aa(a+b)}{(2B - Ab)(x_1 - b) - Bb(a+b)}.$$

Mais $a + b = 0$; donc

$$\frac{x_2 - a}{x_2 - b} = \frac{2B - Aa}{2B - Ab} \cdot \frac{x_1 - a}{x_1 - b}.$$

On aura pareillement

$$\frac{x_3 - a}{x_3 - b} = \frac{2B - Aa}{2B - Ab} \cdot \frac{x_2 - a}{x_2 - b},$$

$$= \left(\frac{2B - Aa}{2B - Ab}\right)^2 \cdot \frac{x_1 - a}{x_1 - b},$$

et en général

$$\frac{x_{n+1} - a}{x_{n+1} - b} = \left(\frac{2B - Aa}{2B - Ab}\right)^n \cdot \frac{x_1 - a}{x_1 - b}.$$

Si on a $x_{n+1} = x_1$, cette relation donne

$$\left(\frac{2B - Aa}{2B - Ab}\right)^n = 1.$$

Par suite
$$\frac{2B - Aa}{2B - Ab} = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}.$$

Mais, A et C étant dans nos hypothèses des quantités essentiellement positives, on a

$$a = -\sqrt{\frac{C}{A}}i, \quad b = +\sqrt{\frac{C}{A}}i,$$

d'où
$$Aa = -\sqrt{AC}i, \quad Ab = +\sqrt{AC}i;$$

et l'on a

$$\frac{2B + \sqrt{AC}i}{2B - \sqrt{AC}i} = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}.$$

Digitized by Google

En égalant les parties réelles et les parties imaginaires, il vient

$$(5) \qquad 4B^2 - AC = (4B^2 + AC)\cos\frac{2k\pi}{n},$$

$$(6) \qquad 4B\sqrt{AC} = (4B^2 + AC)\sin\frac{2k\pi}{n},$$

d'où (7) $$\operatorname{tg}\frac{2k\pi}{n} = \frac{4B\sqrt{AC}}{4B^2 - AC},$$

condition nécessaire et suffisante pour que les relations (5) et (6) soient satisfaites, car le module du premier membre de la relation précédente est égal à l'unité. On tire de (7)

$$\operatorname{tg}\frac{k\pi}{n} = \frac{4B^2 - AC \pm \sqrt{16AC B^2 + (4B^2 - AC)^2}}{4B\sqrt{AC}}$$

c'est à dire $$\operatorname{tg}\frac{k\pi}{n} = \frac{4B^2 - AC \pm (4B^2 + AC)}{4B\sqrt{AC}}.$$

En remplaçant A, B, C par leurs valeurs, il vient

$$\operatorname{tg}\frac{k\pi}{n} = \frac{2r\rho}{\sqrt{(\rho^2 + d^2 - r^2)^2 - 4d^2\rho^2}}$$

et $$\operatorname{tg}\frac{k\pi}{n} = \frac{-\sqrt{(\rho^2 + d^2 - r^2)^2 - 4d^2\rho^2}}{2r\rho}.$$

Ces deux valeurs sont inverses l'une de l'autre et de signes contraires, ce qu'on savait. On peut donc simplement conserver la dernière, et, en ne considérant que la valeur absolue, écrire

$$\operatorname{tg}\frac{k\pi}{n} = \frac{1}{2r\rho}\sqrt{(\rho^2 + d^2 - r^2)^2 - 4d^2\rho^2}.$$

C'est bien la relation donnée.

The Nine-Point Circle.

By Rev. John Wilson, M.A.

I. To find the nine points.

(1) Let M (fig. 20) be the centre of the circle which passes through H, K, L, the middle points of BC, BA, CA.

Draw the diameter HMU.

Join UX, UA, and the three lines forming the median $\triangle$ HKL.

UX is $\perp$ to BC and KL.

Digitized by Google

(2) Let △ HKL be turned through 180° round M, then H, K, L will assume the positions U, V, W;
and WU = and ‖ HL is also = and ‖ AK, the half of AB.

Hence AU is = and ‖ KW.

(3) Now the ∠ LKW is a right angle.

Hence KW and AU are ⊥ KL (and BC),
and therefore ‖ UX.

AUX is therefore a straight line ⊥ to BC.

Similarly BZ, and CY are straight lines, and they intersect in O, the orthocentre;

and H, K, L
U, V, W } are points in the circle.
X, Y, Z

H, K, L are the middle points of the △ ABC;

X, Y, Z the feet of the ⊥^{ars} from the vertices A, C, B on the opposite sides;

U, V, W the points in which these ⊥^{ars} cut the circle.

II. To show that U, V, W, the points in which the ⊥^{ars} from the vertices to the opposite sides cut the medioscribed circle, bisect the segments of the ⊥^{ars} between the orthocentre and the vertices.

In the △ AOB,
KW is drawn through the middle point of AB ‖ to the base;
hence, KW = ½AO.
∴ AU = UO.

Similarly BW = WO and CV = VO.

III. To show that S, the circum-centre, corresponds with O the ortho-centre. Suppose the △ HKL swung round as before.

If LS and HS be drawn ⊥ to the sides, S is the centre of the circumscribed circle.

In the triangles LHS, UWO

LH is ‖ and = UW
HS ‖ UO
LS ‖ WO

Hence the triangles are congruent
and HS = UO.

IV. The line joining S and O is bisected in M.

Digitized by Google

Since HS is ∥ and = UO,
UOHS forms a ∥^{m} whose diagonals mutually bisect.
But UH is bisected in M;
hence SO passes through and is bisected in M.

V. If the median AH be drawn it will cut OS in G so that OG = 2GS.

Through U draw UP ∥ OG;
then UP = ½OG.

AG is bisected in P, hence a line PQ drawn through P ∥ AO bisects OG and is = ½AO = UO = SH.

Hence PQHS is a ∥^{m} whose diagonals mutually bisect.

Hence QG = GS,
or GS = ½OS.

HG is also equal to ½AH.

Hence G is the point of intersection of the medians of the △ ABC.

VI. O, M, G, and S, respectively the ortho-centre, the centre of the medioscribed circle, the centroid and the centre of the circumscribed circle, are collinear.

Sixth Meeting, April 13th, 1888.

W. J. Macdonald, Esq., M.A., F.R.S.E., President, in the Chair.

Extension of a theorem of Abel's in summation to integration.

By George A. Gibson, M.A.

The extension referred to was first given, I believe, by M. Ossian Bonnet in *Liouville's Journal*, vol. xiv., pp. 249 *et seq.* M. Jordan, in his *Cours d'Analyse*, tom. 2, § 82, seems to refer to this article in attributing to M. Bonnet the discovery of the "Second Theorem of the Mean," though he does not explicitly say so. In a course of lectures on "Simple and Multiple Integrals," delivered at Berlin University in the summer of 1885 by Professor Kronecker, which I attended, the "Second Theorem of the Mean" was shown to be a case of Abel's theorem, though I do not remember that Professor Kronecker mentioned M. Bonnet's name in connection with it. I

Digitized by Google

have therefore thought that it might not be without interest to the members of this society to direct attention to the theorem, and to M. Bonnet's connection with it. The proof given is very similar to that in Jordan's *Cours d'Analyse.*

The theorem of Abel in question is Theorem III. of the introduction to his memoir on the Binomial Series (*Collected Works*, vol. I., p. 222). For present purposes we may state the theorem as follows:—If $A_0, A_1, \ldots A_m$ bea series of quantities* such that for all values of r from 1 to m, $A_r - A_{r-1}$ retains the same sign, and if P_r (where $P_r = B_0 + B_1 + \ldots + B_r$, and B_0, B_1, &c., are any quantities whatever) always lies between M and N, then the sum

$$Q = A_0B_0 + A_1B_1 + \ldots + A_mB_m$$

will always lie between MA_0 and NA_0.

We have $P_0 = B_0$, $P_1 - P_0 = B_1$ $\ldots$ $P_m - P_{m-1} = B_m$

$$\therefore Q = A_0P_0 + A_1(P_1 - P_0) + \ldots + A_r(P_r - P_{r-1}) + \ldots + A_m(P_m - P_{m-1})$$
$$= (A_0 - A_1)P_0 + (A_1 - A_2)P_1 + \ldots + (A_{r-1} - A_r)P_r + \ldots \ldots \ldots$$
$$+ (A_{m-1} - A_m)P_{m-1} + A_mP_m.$$

Suppose $M > P_r > N$ and we see that Q lies in value between MA_0 and NA_0 since $A_r - A_{r-1}$ is always of the same sign. We may therefore write $Q = [N + \theta(M - N)]A_0$, where θ is a positive proper fraction.

If we now suppose the As and Bs to be functions of a variable x, we may get some of the ordinary theorems of integration.

Let $A_r = f(x_r)$ where $f(x)$ always varies in the same sense from x to x_m, and let $B_r = \phi(x_r)h_r$ where h_r is infinitesimal.

$\therefore P_r = \int_{x_0}^{x_r} \phi(x)dx$ and M and N are the greatest and least values of $\int_{x_0}^{x_r} \phi(x)dx$ $(r = 1, 2 \ldots m)$

$$\therefore Q = \int_{x_0}^{x_m} f(x)\phi(x)dx = [N + \theta(M - N)]f(x_0) = f(x_0)\int_{x_0}^{\xi} \phi(x)dx$$

where ξ lies between x_0 and x_m. [It is evident that $M + \theta(M - N)$ can be put in the form $\int_{x_0}^{\xi} \phi(x)dx$.]

---

* $A_0, A_1, \ldots \ldots A_m$ are supposed to be positive quantities.

Digitized by Google

This result is very like that given by M. Bonnet.

Again, we have

$$Q = \sum_{r=0}^{r=m} A_0 B_0 = \sum_{r=1}^{r=m} (A_{r-1} - A_r) P_{r-1} + A_m P_m$$

and, as before, $\sum_{r=1}^{r=m} (A_{r-1} - A_r) P_{r-1}$ lies between $M(A_0 - A_m)$ and $N(A_0 - A_m)$

$$\therefore\ Q = [N + \theta(M - N)](A_0 - A_m) + A_m P_m$$

$$\therefore \int_{x_0}^{x_m} f(x)\phi(x)dx = [N + \theta(M - N)](f(x_0) - f(x_m)) + f(x_m)\int_{x_0}^{x_m} \phi(x)dx$$

$$= (f(x_0) - f(x_m))\int_{x_0}^{\xi} \phi(x)dx + f(x_m')\int_{x_0}^{x_m} \phi(x)dx$$

$$= f(x_0)\int_{x_0}^{\xi} \phi(x)dx + f(x_m)\int_{\xi}^{x_m} \phi(x)dx$$

which is the ordinary form of the "Second Theorem of the Mean."

Lastly, we may note that the theorem of "Integration by parts" is virtually given in Abel's theorem, for we have

$$Q = A_0 P_0 + \sum_{r=1}^{r=m} A_r (P_r - P_{r-1}) = \sum_{r=1}^{r=m} (A_{r-1} - A_r) P_r + A_m P_m$$

Let $A_r = f(x_r)$ and $P_r = \phi(x_r)$

$$\therefore \qquad A_{r-1} - A_r = -\frac{df(x_r)}{dx} h_r, \quad P_r - P_{r-1} = \frac{d\phi(x_r)}{dx_r} h_r$$

$$\therefore \quad f(x_0)\phi(x_0) + \int_{x_0}^{x_m} f(x)\phi'(x)dx = f(x_m)\phi(x_m) - \int_{x_0}^{x_m} f'(x)\phi(x)dx$$

$$i.e., \quad \int_{x_0}^{x_m} f(x)\phi'(x)dx = f(x_m)\phi(x_m) - f(x_0)\phi(x_0) - \int_{x_0}^{x_m} f'(x)\phi(x)dx.$$

On the inscription of a triangle of given shape in a given triangle.

By R. E. Allardice, M.A.

§ 1. *To inscribe in a triangle ABC a triangle similar to the triangle DEF, and having its sides parallel to those of DEF.*

In order to inscribe in the triangle ABC (fig. 21), a triangle

Digitized by Google

having its sides parallel to those of DEF, through D, E, F, draw lines parallel to the sides of ABC, and then reduce the figure A'B'C' in the ratio BC : B'C'.

Thus an infinite number of triangles may be inscribed in a given triangle, similar to another given triangle.

A direct construction may also be given, as follows:

In BC (fig. 22) take any point G: draw GH parallel to DE; HK parallel to EF; KL parallel to FD. Then we may easily calculate the ratio A'L : A'G where A' is the vertex of the required triangle that lies in BC.

For,
$$\frac{A'L}{A'G} = \frac{A'L}{C'K} \cdot \frac{C'K}{B'H} \cdot \frac{B'H}{A'G}$$
$$= \frac{LB}{BK} \cdot \frac{KA}{AH} \cdot \frac{HC}{CG}$$ which is known.

[Dr Mackay suggests a modification of this method, depending on the fact that A, A' and the point of intersection of KL and HG are collinear.]

The theorems of §§ 2, 3, and 4 are required further on in this paper.

§ 2. *To find the condition that the perpendiculars to the sides of a triangle ABC, drawn at the points D, E, F, in the sides, be concurrent.*

Let the perpendiculars at D, E, F, (fig. 23) meet in the point O.

Since $AO^2 - BO^2 = AF^2 - BF^2$, the necessary and sufficient condition for concurrence is obviously

$$AF^2 - BF^2 + BD^2 - CD^2 + CE^2 - AE^2 = 0$$

or,
$$AF^2 + BD^2 + CE^2 = BF^2 + CD^2 + AE^2.$$

§ 3. *If one triangle is inscribed in another triangle, and if the perpendiculars from the vertices of one triangle on the sides of the other triangle are concurrent, then the perpendiculars from the vertices of the second triangle on the sides of the first are also concurrent.*

Let DEF (fig. 24) be inscribed in ABC; and let the perpendiculars at D, E, F, to the sides BC, CA, AB, meet in the point O; then the perpendiculars from A, B, C, on the sides of EF, FD, DE, will also meet.

Let the perpendiculars from A, B, C, meet the sides of DEF in A', B', C'.

Digitized by Google

Since the perpendiculars at D, E, F, are concurrent,

$$\therefore \quad AF^2 - BF^2 + BD^2 - CD^2 + CE^2 - AE^2 = 0.$$

But $AF^2 - AE^2 = A'F^2 - A'E^2$, etc.

$$\therefore \quad A'F^2 - B'F^2 + B'D^2 - C'D^2 + C'E^2 - A'E^2 = 0;$$

which proves the theorem.

§ 4. *The lines joining the points O and P of last paragraph to the vertices are equally inclined to the bisectors of the angles of the triangle.*

Let O and P (fig. 24) be the two points.

Then, $\angle FAO = \angle FEO =$ complement of $\angle FEA = \angle EAP$.

In the recent geometry of the triangle, lines equally inclined to the bisector of an angle are called *isogonal* lines; and the points of concurrence of one set of three lines passing through the vertices and of the three isogonal lines are called *inverse* points. Thus OA and PA, OB and PB, OC and PC, are pairs of isogonal lines; and O and P are inverse points.

§ 5. The problem that first suggested itself, and that led to this paper, was as follows:

To find a point P within a triangle such that the images Q, R, S, of P in the three sides shall be the vertices of an equilateral triangle.

This is obviously the same problem as the following:

To inscribe, in a given triangle, an equilateral triangle, such that the perpendiculars to the sides of the given triangle, drawn through the vertices of the equilateral triangle, shall be concurrent.

First Method.

This problem may be solved by finding the point inverse (in the sense mentioned above) to the point of intersection of the perpendiculars through the vertices of the required equilateral triangle.

The corresponding point is, in fact, the point of concurrence of the circles circumscribing the equilateral triangles, described on the sides of the given triangle.

Suppose DEF (fig. 24) to be equilateral; then

$$\angle PAC = \angle OAF = \angle OEF,$$
$$\angle PCA = \angle OCB = \angle OED;$$
$$\therefore \quad \angle PAC + \angle PCA = \angle DEF = 60°, \quad \therefore \quad \angle APC = 120°.$$

Thus the problem is solved.

§ 6. Eight systems of three circles may be obtained as the circles

Digitized by Google

circumscribing equilateral triangles described on the sides of a given triangle. The question naturally arises, In how many of these systems do the circles concur in one real point? We shall show that this happens only in the case of two of the systems.

First Case. Consider a triangle ABC with the angle C greater than 120°.

There is obviously a point of concurrence on each of the arcs of 120° described on AB, and none on either of the arcs of 60° described on AB.

Second Case. Consider a triangle ABC, with no angle greater than 120°.

There is always one point of concurrence of circles within the triangle, and one without the triangle. If the triangle has only one angle greater than 60°, the point of concurrence that lies outside the triangle is on the arc of 120° described on the greatest side of the triangle; while if the triangle has two angles greater than 60°, this point of concurrence is on the arc of 120° described on the shortest side of the triangle.

We may also show analytically, by getting the equations to the circles, that in only two of the systems do the three circles meet in one point.

The equation to the circle circumscribing the equilateral triangle described *externally* on the side BC is

$$(a\beta\gamma + b\gamma a + ca\beta)\sin 60° - \gamma(aa + b\beta + c\gamma)\sin(60° + A) = 0\,; \qquad (1)$$

while the equation to the circle circumscribing the equilateral triangle described *internally* on the same side (that is, so that the remaining vertex of the equilateral triangle and the vertex A are on the same side of BC), is

$$(a\beta\gamma + b\gamma a + ca\beta)\sin 60° - \gamma(aa + b\beta + c\gamma)\sin(60° - A) = 0. \qquad (2)$$

Now, it may be shown that the circle (1) and the other two corresponding circles meet in a point, and that the same is true of the circle (2) and the two other circles corresponding to it; and that these are the only two systems containing three concurrent circles.

§ 7. Another construction may be given for the points P of § 6, that is, for the points of concurrence of the circles circumscribing the equilateral triangle described on the sides of the given triangle.

It may, in fact, easily be shown that if equilateral triangles BCD, CAE, ABF, (fig. 25) are described on the sides of the triangle ABC,

Digitized by Google

the vertices A and D being on opposite sides of BC, B and E on opposite sides of CA and C and F on opposite sides of AB, then the lines AD, BE, CF are concurrent; and that the point of concurrence is the point of concurrence of the circles circumscribing the triangles BCD, CAE, ABF.

The same result follows if the equilateral triangles are described so that A and D are on the same side of BC, B and E on the same side of CA and C and F on the same side of AB.

§ 8. Second Method.

Let ABC (fig. 24) be the given triangle, DEF the required inscribed equilateral triangle; then, by the formula for the chord of a circle, in terms of the angle it subtends at the circumference and the diameter of the circle,

$$EF = OA\sin A = FD = OB\sin B = DE = OC\sin C;$$

$$\therefore \quad OA : OB : OC = 1/\sin A : 1/\sin B : 1/\sin C = 1/a : 1/b : 1/c.$$

Hence the point O may be found as follows:—

Let the bisectors of the angles at A meet BC in D and D'; on DD' as diameter describe a circle. The two points of intersection of this circle, and the circles obtained by taking the other two sides instead of the side BC, are the points required.

§ 9. *Analytical Investigation.*

In fig. 24, let $OD = \alpha$, $OE = \beta$, $OF = \gamma$, $DE = EF = FD = l$; then

$$OE^2 + OF^2 + 2OE.OF\cos A = l^2;$$

or,
$$\beta^2 + \gamma^2 + 2\beta\gamma\cos A = l^2; \qquad (1)$$

similarly,
$$\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B = l^2; \qquad (2)$$

$$\alpha^2 + \beta^2 + 2\alpha\beta\cos C = l^2. \qquad (3)$$

On subtracting the second of these equations from the first we get the equation to one of the circles of § 7, namely,

$$\beta^2 - \alpha^2 + 2\gamma(\beta\cos A - \alpha\cos B) = 0; \qquad (4).$$

and we get the other two by taking the other two differences.

Equation (4) may be expressed in the form

$$(b^2 - c^2)(a\beta\gamma + b\gamma\alpha + c\alpha\beta) + a(a\alpha + b\beta + c\gamma)(c\beta - b\gamma) = 0. \qquad (5)$$

The radical axis of the three circles given by the equations (1), (2), (3), is

$$bc(b^2 - c^2)\alpha + ca(c^2 - a^2)\beta + ab(a^2 - b^2)\gamma = 0;$$

or,
$$\alpha\sin(B - C) + \beta\sin(C - A) + \gamma\sin(A - B) = 0.$$

A simple geometrical construction may be given for this line,

Digitized by Google

namely (fig. 26) make $\angle ABQ = \angle ACQ = A$; let AQ meet BC in P; then P is the point where the radical axis meets BC.

The equation to the radical axis of the circle of equation (5) and the circumcircle of the triangle is obviously $c\beta - b\gamma = 0$. It may easily be shown by means of this equation that if the radical axis meets the side AB in O, then $AO : OB = b^2 : a^2$. This may also be proved without the use of the above equations, by calculating the segments AO and OB. It follows from this result that AO is one of the symmedians of the triangle.

§ 10. Generalization.

To find a point O (fig. 27) such that, if OD', OE', OF', be the perpendiculars from O on the sides of ABC, the triangle D'E'F' shall be similar to any given triangle DEF.

On AB, BC, CA, describe the triangles ABC', A'BC, AB'C, similar to the triangle DEF (the triangles are named as they correspond to DEF); and let all these triangles be described externally or all internally on the sides of ABC. Then in both cases the circles circumscribing these triangles will be concurrent (in a point P); and the point inverse to this point of concurrence will be the point O required. The proof is almost identical with that given before for the special case when DEF is equilateral.

The point P may also be obtained as the point of concurrence of the lines AA', BB', CC'.

It should be noticed that, according to the construction given above, the point corresponding to D will lie in BC, the point corresponding to E in CA, and the point corresponding to F in AB. By making the triangles ABC', AB'C, A'BC, similar to EFD, the vertices corresponding in this order, we may make the vertex corresponding to E lie in BC, that corresponding to F lie in CA, and that corresponding to E in AB; and by making other variations in the correspondence of the similar triangles, we may make the vertices corresponding to those of DEF lie in whichever sides of ABC we choose.

An analytical investigation, similar to that of § 9, may also be given.

Digitized by Google

Pappus on the Progressions.

By J. S. MACKAY, LL.D.

[The present paper is a translation of the second part of the third book of Pappus's *Mathematical Collection*. Pappus's date is uncertain, but 300 A.D. may be taken as an approximation to it.

Throughout the translation I have used the word "progression" as a rendering of the Greek μεσότης, which has no English equivalent. The only other alternative was to employ the term *mediety*, from the Latin *medietas*.

The account of the various progressions given by Nicomachus, in his *Arithmetical Introduction*, differs somewhat from that of Pappus. I hope to have something to say about Nicomachus in a future paper.]

The second problem was this:

Figure 28.

Some other person said that the three progressions could be obtained in a semicircle thus. He described a semicircle ABC whose centre was E; taking any point D in AC, and drawing DB at right angles to EC, he joined EB, and from D drew DF perpendicular to it. The three progressions, he maintained, were exhibited in a simple manner in the semicircle; for EC was the arithmetical mean, DB the geometrical, and BF the harmonical.

That BD is the mean of AD, DC in a geometrical proportion, and EC the mean of AD, DC in an arithmetical progression is evident. For $AD : DB = DB : DC$;

and $AD : AD = AD - AE : EC - CD,$

$= AD - EC : EC - CD.$

But how BF is the mean of the harmonical progression, or of what straight lines it is the mean, he has not said; but only that it is the third proportional of EB, BD, not knowing that the harmonical progression is formed from EB, BD, BF when they are in a geometrical proportion*. For it will be shown by us later on that $2EB + 3DB + BF$ is the greater extreme, $2DB + BF$ is the mean, and $DB + BF$ is the least extreme of the harmonical progression.

But first, in order that we may more fully discuss the proposed demonstration, one must treat of the three progressions, and, after

---

* This censure from Pappus seems to be quite undeserved.

Digitized by Google

this, of the progressions in a semicircle, then of the three others which, according to the ancients, are opposed to these, and lastly, in accordance with the views of more recent geometers, of the four devised by them; and how it is possible to find by means of a geometrical proportion each of the ten progressions.

The three progressions.

A progression differs from a proportion in this, that every proportion is a progression, but not conversely. For there are three progressions—the arithmetical, the geometrical, the harmonical.

A progression is called arithmetical when there are three terms and the mean exceeds one of the terms by the same as it is exceeded by the remaining term, as 6 with reference to 9 and 3; or when the first term is to itself as the first difference to the second.

A progression is called geometrical, that is a proportion strictly, when the mean is to one of the terms as the remaining term is to the mean, as 6 with reference to 12 and 3; and otherwise, when the first term is to the second as the first difference to the second.

A progression is harmonical when the mean exceeds one of the extremes by the same fraction as it is exceeded by the remaining extreme, as 3 with reference to 2 and 6; or when the first term is to the third as the first difference is to the second difference.

These things having been laid down, we shall find the three progressions together in the five minimum straight lines, after premising the following.

First, having given the straight lines AB, BC, let it be proposed to find the mean according to a geometrical proportion.

Figure 29.

Let CD be drawn at right angles, and let AB be bisected at E. With centre E let a circle be described through B cutting CD at D; join BD and cut off BF equal to it. BF is the required mean.

If DA be joined, it contains a right angle with BD, because both BE and EA are equal to DE. Now the angle at C is right; therefore the triangle ABD is equiangular to the triangle BCD, and accordingly the sides about their common angle at B are proportional. Hence AB : DB = BD : BC, and BD or BF is the mean between AB and BC.

Given AB and BF let it be proposed to find the less extreme.

Digitized by Google

Figure 30.

Let AB be bisected at E; with centre E let a circle be described through B, and let this circle be cut at D by a circle described through F with centre B. Let a perpendicular DC be drawn; then BC is the third proportional to AB, BF.

The proof is similar to that regarding the mean.

Given FB, BC let it be proposed to find the greater extreme.

Figure 31.

Let CH be drawn at right angles, and with centre B let a circle described through F cut CH at H. Join BH, and draw AH at right angles to it. Then AB is the third proportional to CB, BF.

This is obvious from what has been proved before.

Figure 32.

Again, let there be two straight lines AB, BC, and let DAE be at right angles to AB, so that AD is equal to AE. Let BD, ECF be joined, and from F let FG be drawn perpendicular to CB. Then AB : BG = AB – BC : CB – BG.

For AB : BG = DA : FG
= AE : FG, since AE = AD;
= AC : CG, on account of the triangles ACE, CFG being equiangular.

Now AC = AB – BC, and CG = CB – BG;
therefore AB : BG = AB – BC : CB – BG.

Figure 32.

But if the extremes AB, BG be given and we seek the mean, join BD, and from G draw FG at right angles. From F to E draw FCE, and we shall have CB the mean between AB and BG.

The proof is obvious.

Figure 33.

Given EB, BC we shall find the greater extreme by drawing from E, DEF at right angles, making DE = EF, joining BF, DC, and producing them to G.

For GH the perpendicular drawn from G to BC produced will cut off HB equal to what is sought.

Again, given two straight lines AB, C, of which AB is the greater, we shall find the equidifferent mean thus.

Digitized by Google

Figure 34.

Make DB = C, bisect DA at E, and make F = EB. It is obvious that F is the straight line sought.

Similarly if F, C be given, by adding their difference to F we shall have a straight line equal to AB.

Again, if AB, F be given, their difference subtracted from F will give C the third.

Figure 35.

If therefore F be the equidifferent mean of AB, C, the straight lines AB, F, C will form an arithmetical progression. As F is to C so make C to G; then the straight lines F, C, G will form a geometrical progression, that is, a proportion strictly. And if, according to what has been proved before, having two straight lines C, G, the greater of which is C, we make H such that

$$C : H = C - G : G - H,$$

then the straight lines C, G, H will form a harmonical progression. Now AB : C = C : H, AB and C being the extreme terms in the arithmetical progression, and C and H in the harmonical; there will therefore be five minimum straight lines containing the three progressions [and these may be incommensurable with one another].

Now let it be proposed to form the three progressions with the minimum five numbers, and according to what are called multiple, superparticular, and other ratios, unity being supposed indivisible. When the ratio of AB to C is 2, for instance, the minimum numbers which effect what is proposed will be 12, 9, 6, 4, 3; when the ratio is 3, the minimum numbers will be 18, 12, 6, 3, 2. And it is evident how with other ratios also, the minimum numbers for the three progressions must be found. Now if one should wish to express separately each of the progressions, that is clear from what has been previously written; the three terms of the arithmetical progression being in the minimum numbers 3, 2, 1, of the geometrical 4, 2, 1, and the numbers which, according to the given ratio, are fundamental being transformed into equimultiples and superparticulars and the rest. For example, if AB has to C the ratio of 2 to 1, instead of 2 we shall put 4, and instead of 1 we shall put 2. And since the mean between these must exceed and be exceeded by the same amount, the straight line F consists of 3 units. Now the ratio of F to C is that of 3 to 2, and if the ratio of C to G be made equal to it,

Digitized by Google

the problem is not done, because unity remains indivisible. Let everything then be tripled, and 12 is obtained for 4, 9 for 3, 6 for 2. The straight line G then becomes one of 4 units, and H manifestly of 3, and the numbers for the three progressions are 12, 9, 6, 4, 3.

So much, then, concerning the three progressions according to the ancients. Thence it is evident that it is possible to exhibit the three progressions together in a semicircle in the minimum six straight lines.

Figure 36.

Let a semicircle be described having BD perpendicular, and EB a radius, and DF perpendicular to EB. Through B draw HG touching the circle, produce EC to G, make BH equal to BG, and join DKH. Then in the harmonical progression EK is the mean between BE and EF, the greatest term being BE and the least EF.

Since the angles at B and F are right, DF is parallel to HG, and the triangle EBG is equiangular to the triangle EFD, and the triangle BHK to the triangle FKD;

therefore $BE : EF = GB : FD,$

$= HB : FD,$ because $BG = BH;$

$= BK : KF.$

Now $BK = BE - EK, \quad KF = KE - EF;$

therefore $BE : EF = BE - EK : KE - EF.$

The straight lines BE, EK, EF then contain the harmonical progression, the mean being EK, the greatest BE, and the least EF. And AD, EC, CD were shown to contain the arithmetical progression, AD, BD, DC the geometrical. The three progressions therefore have been arranged in a semicircle.

Since Nicomachus,* the Pythagorean, and some others have spoken not only of the first three progressions, which are the most useful in the study of ancient authors, but also of the other three which were in vogue among the ancients, and since, in addition to these six, other four have been invented by more recent writers, we shall endeavour to speak of these somewhat carefully (?), following, however, the older writers who began from the greater term. . . . . [The Greek text is here corrupt.]

For when the third term is to the first as the excess of the first term is to that of the second, they call the progression contra-harmonical.

* In his *Arithmetical Introduction*. Nicomachus's date is about 100 A.D.

Digitized by Google

But when the third term is to the second as the excess of the first is to that of the second, the progression is called the fifth and contro-geometrical, for some name it so.

When the second term is to the first as the excess of the first is to that of the second, the progression is called the sixth. It also is called contra-geometrical, because in the sequence of the ratios the order is reversed. Thus, according to them, there are six progressions.

By more recent writers, as we said, other four have been found, in some respect useful, and their discoverers employ their own definitions. For they call the excess of the first term above the second the first difference, that of the second above the third the second difference, and that of the first above the third the third difference, the greatest term being understood and spoken of, as we explained at the outset, as the first, the mean as the second, and the least as the third.

When the third difference is to the first as the second term to the third, they call the progression the seventh.

But, while the ratio of the differenees remains the same, if it be as the first term to the second, they call the progression the eighth.

If the third difference be to the first as the first term to the third, they call the progression the ninth.

If the third difference be to the second as the second term to the third, they call the progression the tenth.

Having laid down these definitions, we shall explain the origins of the ten progressions, as we said, by means of a geometrical proportion.

The geometrical progression then, since it takes its first origin from equality, will constitute both itself and the other progressions, showing, as saith the divine Plato, that the nature of proportion is the cause of the harmony of all things, and of a rational and ordered creation. For he says that there is one bond of all the sciences. Now the cause of creation and the bond by which all created things are held together is the divine nature of proportion. The constitution of the ten progressions will be shown by means of the geometrical proportion, the following being premised.

Let there be three terms A, B, C proportional, and let $D = A + 2B + C$, $E = B + C$, $F = C$, then the terms D, E, F are proportional.

Since $A : B = B : C$,

Digitized by Google

by composition $A+B:B=B+C:C$;
therefore the sum of the antecedents is to the sum of the consequents in the same ratio,
that is, $A+2B+C:B+C=B+C:C$.
Now $D=A+2B+C$, $E=B+C$, and $F=C$;
therefore $D:E=E:F$.

Hence if A, B, C be supposed equal, D, E, F will be in a double proportion; for $A+2B+C=2(B+C)$ and $B+C=2C$. But if A, B, C be supposed to be in a double proportion and A to be the greatest term among them, D, E, F will be in a triple proportion; and if A be the least term, D, E, F will be in sesquialterate proportion. For if $A=2B$, then $A+B=3B$; and if $A=\frac{1}{2}B$, then $A+B=\frac{3}{2}B$.
And so from the succeeding ratios, the numbers which follow, both multiples and superparticulars, will be found.

And again, if A, B, C were units, the geometrical progression formed by D, E, F would be said to be in minimum numbers 4, 2, 1.

[The translators of Pappus, Commandinus, and Hultsch, in the belief that a lacuna exists here in the Greek text, have inserted a proposition showing how the arithmetical progression is constituted by means of a proportion.

Commandinus puts

$$D=2A+2B+C,\ E=A+B+C,\ F=C,$$

and finds the minimum numbers 5, 3, 1. This, however, does not agree with the entries for the arithmetical progression in the table at the end, as given in his edition of Pappus.

Hultsch puts

$$D=2A+3B+C,\ E=A+2B+C,\ F=B+C,$$

and finds the minimum numbers 6, 4, 2. These are the numbers given for the arithmetical progression in the tables of all the MSS. of Pappus which I have examined.]

The harmonical progression is thus constituted by means of a proportion.

Let three terms A, B, C be supposed proportional,
and let $D=2A+3B+C$, $E=2B+C$, $F=B+C$;
then D, E, F form the harmonical progression.

Since A, B, C are proportional,
therefore $2A+B:B=2B+C:C$.
Taking the sum of the antecedents and the sum of the consequents,

Digitized by Google

$$2A+3B+C : B+C = 2A+B : B,$$

that is $$D : F = 2A+B : B.$$

Now $$D-E = 2A+B, \text{ and } E-F = B;$$

and when $$D : F = D-E : E-F,$$

the progression is harmonical.

And it is evident, if A, B, C be supposed to be units, that the harmonical progression is constituted in minimum numbers 6, 3, 2.

The contra-harmonical progression is thus constituted from a proportion.

Let the terms A, B, C be supposed proportional,

and let $D = 2A+3B+C$, $E = 2A+2B+C$, $F = B+C$;

then D, E, F form the said progression.

For again, similarly to what has been shown before,

$$D : F = 2A+B : B.$$

And $$E-F = 2A+B, \; D-E = B;$$

therefore $$F : D = D-E : E-F,$$

which is what characterises the contra-harmonical progression.

And it is evident, if A, B, C be supposed units, that the progres sion is constituted in minimum numbers 6, 5, 2.

The fifth progression is thus constituted from a proportion.

Let the three terms A, B, C be supposed proportional,

and let $D = A+3B+C$, $E = A+2B+C$, $F = B+C$;

then D, E, F are in the fifth progression.

Since, on account of the proportion,

$$A+B : B = B+C : C$$

taking the sum of the antecedents and the sum of the consequents,

$$A+2B+C : B+C = A+B : B,$$

that is $$E : F = A+B : B.$$

Now, $$E-F = A+B, \text{ and } D-E = B;$$

therefore $$F : E = D-E : E-F,$$

which is what happens in the fifth progression.

And if A, B, C be supposed units, the progression would be said to be in minimum numbers 5, 4, 2.

The sixth progression is thus constituted from a proportion.

Let the proportion of the terms A, B, C be the same,

and let $D = A+3B+2C$, $E = A+2B+C$, $F = A+B-C$;

then D, E, F form the proposed progression.

Since, on account of the proportion,

Digitized by Google

$$A+2B : A+B = B+2C : B+C;$$

taking the sum of the antecedents and the sum of the consequents,

$$A+3B+2C : A+2B+C = B+2C : B+C,$$

that is $$D : E = B+2C : B+C.$$

Now $$E-F = B+2C, \text{ and } D-E = B+C;$$

therefore $$E : D = D-E : E-F,$$

so that D, E, F form the sixth progression.

And if A, B, C be supposed units, it is similarly constituted in minimum numbers 6, 4, 1.

[Here, also, a lacuna has been presumed to exist in the Greek text by Pappus's commentators.

Commandinus puts

$$D = A+2B+2C,\ E = A+B+C,\ F = B+C,$$

and finds the minimum numbers 5, 3, 2. This again does not agree with the entries for the seventh progression in the table, as given in his edition.

Hultsch puts

$$D = A+B+C,\ E = A+B,\ F = C,$$

and finds the minimum numbers 3, 2, 1.

I have given the table at the end, which is much corrupted in the MSS., as it exists in Hultsch's edition, vol. I., pp. 102-103, although I have long entertained some suspicion of its genuineness, as well as of the need for filling up the presumed lacunae.]

The eighth progression is thus constituted from a proportion.

Let the three terms A, B, C be supposed proportional,

and let $$D = 2A+3B+C,\ E = A+2B+C,\ F = 2B+C;$$

then D, E, F are according to the eighth progression.

Since, on account of the proportion,

$$2A+B : A+B = 2B+C : B+C;$$

taking the sum of the antecedents and the sum of the consequents,

$$2A+3B+C : A+2B+C = 2A+B : A+B,$$

that is $$D : E = 2A+B : A+B.$$

Now $$D-F = 2A+B, \text{ and } D-E = A+B;$$

therefore $$D : E = D-F : D-E,$$

which constitutes the eighth progression.

And if A, B, C be supposed units, it would be said to be in minimum numbers 6, 4, 3.

The ninth progression is thus constituted from a proportion.

Let A, B, C be supposed proportional,

Digitized by Google

and let $D = A + 2B + C$, $E = A + B + C$, $F = B + C$;
then D, E, F contain the ninth progression.

Since $A + B : B = B + C : C$;
taking the sum of the antecedents and the sum of the consequents,

$$A + 2B + C : B + C = A + B : B,$$

that is $D : F = A + B : B$.
Now $D - F = A + B$, and $D - E = B$;
therefore $D : F = D - F : D - E$,
which is the characteristic of the ninth progression.

And if A, B, C be similarly supposed units, the minimum numbers 4, 3, 2 contain the progression.

The tenth progression is thus constituted from a proportion.

Again let the three A, B, C be proportional,
and let $D = A + B + C$, $E = B + C$, $F = C$;
then D, E, F are according to the tenth progression.

For $B + C : C = A + B : B$,
that is $E : F = A + B : B$.
Now $D - F = A + B$, and $E - F = B$;
therefore $E : F = D - F : E - F$,
which happens in the tenth progression.

And if A, B, C be supposed units, the minimum numbers 3, 2, 1 form the progression.

For the sake of convenience there are set out the successive numbers by which each term of the proportion is multiplied so as to form each progression, and beside them are placed the minimum numbers containing the progressions. For instance, in the table of the sixth progression the first row 1, 3, 2 means this, that the first term of the proportion taken once, the second thrice, and the third twice, complete the first term of the progression; the second row of the table 1, 2, 1 means that the first term of the proportion taken once, the second twice, and the third once, complete the second term of the progression. The third row of the table in the remaining progressions is composed simply as has been described; exceptionally, however, in this progression the row 1, 1, 1 signifies, as has been said before, that the third term of the progression is obtained from the difference by which the first term of the proportion taken once, and the second taken once, exceed the third term taken once. In the third part of the table the numbers 6, 4, 1 contain the progression itself. Let similar things be understood regarding the remaining tables.

Digitized by Google

Progressions.	Coefficients of the Terms A, B, C.	Three Minimum Numbers containing the Progressions.
Arithmetical	2 3 1 1 2 1 1 1	6 4 2
Geometrical	1 2 1 1 1 1	4 2 1
Harmonical	2 3 1 2 1 1 1	6 3 2
Contra-harmonical	2 3 1 2 2 1 1 1	6 5 2
Fifth	1 3 1 1 2 1 1 1	5 4 2
Sixth	1 3 2 1 2 1 1 1	6 4 1
Seventh	1 1 1 1 1 1	3 2 1
Eighth	2 3 1 1 2 1 2 1	6 4 3
Ninth	1 2 1 1 1 1 1 1	4 3 2
Tenth	1 1 1 1 1 1	3 2 1

Digitized by Google

Seventh Meeting, May 11th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

"Vortex Rings in a Compressible Fluid."

By C. CHREE, M.A.

In a paper recently printed in the Society's *Proceedings*, I considered the effect of compressibility in the fluid on the motion of straight vortices; the present paper treats of circular vortex rings in a compressible fluid. The circle passing through the centres of the circular cross sections of the vortex filament will be called the "circular axis," and the perpendicular to the plane of the circular axis through its centre, the "axis" of the vortex. In the notation employed, a denotes the radius of the circular axis, and e that of the cross section of the filament, while ω represents vorticity, and ρ density. It is also convenient to denote the area of the cross section—*i.e.*, πe^2, by σ. Following Helmholtz, it will be supposed that e/a is always very small, and that the cross section is truly circular. Certain small inconsistencies in the ordinary theory following from this last assumption will be pointed out, though they do not seem seriously to affect the general applicability of the results. The axis of the vortex ring is taken as axis of z, and z, r, θ are the ordinary cylindrical co-ordinates. It is also convenient to denote by r' the distance of a point from the circular axis of a ring, and by ψ the inclination of this distance to the plane of the circular axis. The effects of the vorticity and variation in density may be considered separately.

The components of vorticity at any point of the ring are $\xi = -\omega\sin\theta$, $y = \omega\cos\theta$. There is obviously symmetry about oz, and the velocity at any point can be resolved into w parallel to oz and u along the perpendicular on oz. Since e/a is very small, we shall, following the common practice in calculating the velocity, regard the

Digitized by Google

vortex filament as concentrated in the circular axis, and so in Lamb's formulæ* replace the vortex element $dxdydz$ by $\frac{ma}{\omega}d\theta$; where $m, = \pi c^2\omega$, is the strength of the vortex. Supposing the origin taken in the instantaneous position of the centre of the circular axis, we find from these formulæ for the velocity in the fluid at the point r, θ, z outside the vortex

$$w = \frac{ma}{2\pi}\int_0^{2\pi} \frac{(a - r\cos\theta)d\theta}{(z^2 + a^2 + r^2 - 2ar\cos\theta)^{\frac{3}{2}}} \quad \ldots \quad \ldots \quad (1),$$

$$u = \frac{ma}{2\pi}\int_0^{2\pi} \frac{z\cos\theta d\theta}{(z^2 + a^2 + r^2 - 2ar\cos\theta)^{\frac{3}{2}}} \quad \ldots \quad \ldots \quad (2).$$

We may at once transform (1) into

$$w = -\frac{2ma}{\pi}\frac{d}{da}\left[\frac{1}{\sqrt{z^2 + (r+a)^2}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}\right] \quad \ldots \quad \ldots \quad (3),$$

$$\text{where } k^2 = \frac{4ra}{z^2 + (r+a)^2},\ 2\phi = \pi - \theta \quad \ldots \quad \ldots \quad (4).$$

Thus if, as usual, F_1 denote the complete elliptic integral of the first order,

$$w = -\frac{2ma}{\pi}\frac{d}{da}\left[\{z^2 + (r+a)^2\}^{-\frac{1}{2}}F_1(k)\right] \quad \ldots \quad \ldots \quad (5).$$

When k is nearly unity, an approximate value is $F_1(k) = \log(4/k_1)$,

$$\text{where } k_1{}^2 = 1 - k^2 = \frac{z^2 + (r-a)^2}{z^2 + (r+a)^2} \quad \ldots \quad \ldots \quad (6).$$

Thus, for points near the surface of the filament, where z and $r - a$ are both very small, an approximate value is

$$w = -\frac{2ma}{\pi}\frac{d}{da}\left[\{z^2 + (r+a)^2\}^{-\frac{1}{2}}\log\left\{4\left(\frac{z^2 + (r+a)^2}{z^2 + (r-a)^2}\right)^{\frac{1}{2}}\right\}\right].$$

This gives

$$w = \frac{2ma}{\pi}\{z^2 + (r+a)^2\}^{-\frac{3}{2}}\left[(r+a)\log\left\{4\left(\frac{z^2 + (r+a)^2}{z^2 + (r-a)^2}\right)^{\frac{1}{2}}\right\} - \frac{2r(z^2 + r^2 - a^2)}{z^2 + (r-a)^2}\right] \quad \ldots \quad \ldots \quad (7).$$

For points outside, but in contact with the surface of the filament, we have $r = a + e\cos\psi, z = e\sin\psi$. Substituting these values in (7) and

* "Motion of Fluids"—Equations (15), p. 152.

Digitized by Google

retaining the principal terms, we get for the velocity in the fluid just outside the filament

$$w = \frac{m}{2\pi a}\left\{\log\frac{8a}{e} - 1 + \cos^2\psi - \frac{2a\cos\psi}{e}\right\} \quad \ldots \quad \ldots \qquad (8).$$

The first and last are much the most important terms.

The most important term omitted is $-\frac{me\cos\psi}{2\pi a^2}\log\frac{8a}{e}$.

From (2), using k and ϕ in the same sense as before, we easily find

$$u = \frac{m}{\pi r}\frac{d}{dz}\left[\{z^2 + (r+a)^2\}^{\frac{1}{2}}\mathrm{E}_1(k) - \frac{z^2 + r^2 + a^2}{\{z^2 + (r+a)^2\}^{\frac{1}{2}}}\mathrm{F}_1(k)\right] \quad \ldots \quad (9);$$

where E_1 is the complete elliptic integral of the second order.

For points near the surface of the filament, we saw that approximately $\mathrm{F}_1(k) = \log(4/k_1)$;

also
$$\mathrm{E}_1(k) = (1 - k^2)\left\{\mathrm{F}_1(k) + \frac{kd}{dk}\mathrm{F}_1(k)\right\} \quad \ldots \quad \ldots \quad (10).*$$

From its approximate value we get $k\frac{d\mathrm{F}_1(k)}{dk} = \frac{k^2}{k_1^2} = \frac{4ar}{z^2 + (r-a)^2}$;

thus an approximate value is

$$\mathrm{E}_1(k) = \frac{z^2 + (r-a)^2}{z^2 + (r+a)^2}\left[\log\left\{4\left(\frac{z^2 + (r+a)^2}{z^2 + (r-a)^2}\right)^{\frac{1}{2}}\right\} + \frac{4ar}{z^2 + (r-a)^2}\right] \quad \ldots \quad (11).$$

Substituting these values in (9), carrying out the differentiations and reducing, we obtain the approximate value

$$u = \frac{2maz}{\pi}\left\{z^2 + (r+a)^2\right\}^{-\frac{3}{2}}\left[\log\left\{4\left(\frac{z^2 + (r+a)^2}{z^2 + (r-a)^2}\right)^{\frac{1}{2}}\right\} - 2 + \frac{4ar}{z^2 + (r-a)^2}\right] \quad \ldots \quad \ldots \quad (12)$$

Using the same notation as before, and retaining the principal terms, we find for the velocity in the fluid just outside the filament

$$u = \frac{m}{\pi e}\sin\psi - \frac{m}{2\pi a}\sin\psi\cos\psi \quad \ldots \quad \ldots \qquad (13).$$

The most important term omitted is $\frac{me\sin\psi}{4\pi a^2}\log\frac{8a}{e}$.

From (8) and (13) we see that the velocity in the fluid just outside the vortex is composed of the two components

$$w = \frac{m}{2\pi a}\left\{\log\frac{8a}{e} - 1\right\} \quad \ldots \quad \ldots \qquad (14),$$

---

* Cayley's "Elliptic Functions."

Digitized by Google

parallel to the axis oz, and

$$\tau = \frac{m}{\pi e}\left(1 - \frac{e}{2a}\cos\psi\right) \quad \ldots \quad \ldots \qquad (15),$$

tangential to the surface of the filament in the plane containing oz.

The former component (14) is a motion *en masse*, the same at every point of the surface of the filament, and is shared by the vortex and the fluid bordering on it; the latter (15) represents the velocity of circulation of the fluid round the circular axis. This velocity is slightly greater on the concave, or inner, side of the filament, and less on the convex side than in the case of a straight vortex of the same strength and cross section.

This is one of the small inconsistencies already referred to; for if the vorticity ω be constant throughout the filament and the section truly circular, the velocity of circulation in the fluid just inside the surface of the filament must be ωe, while from (15) the fluid just outside has its velocity of circulation $= \omega e\left(1 - \frac{e}{2a}\cos\psi\right)$. There is thus a very slight absence of continuity in the motion on crossing the surface. In a perfectly frictionless fluid this may seem of absolutely no importance, but the following reasoning shows that an inconsistency of a precisely similar character exists in the hypothesis that ω can be constant throughout a truly circular section.

Using r' in the sense already indicated, let us not assume ω to be constant, but still suppose the velocity perpendicular to r'. Consider the elementary ring formed by the revolution about oz of the element $r'd\psi dr'$ of the cross section of the filament. The volume of this ring is $2\pi(a + r'\cos\psi)r'd\psi dr'$, and the areas of the two surfaces through which alone flow takes place are each $2\pi(a + r'\cos\psi)dr'$. The velocity is normal to these surfaces and equal to $\omega r'$. Thus, by the equation of continuity, if the fluid be incompressible, we get

$$\frac{d}{d\psi}\left[\left(1 + \frac{r'}{a}\cos\psi\right)\omega r'\right] + \frac{d}{dt}\left[\left(1 + \frac{r'}{a}\cos\psi\right)r'\right] = 0.$$

Since $\frac{d\psi}{dt} = \omega$, this gives at once

$$\omega\left(1 + \frac{r'}{a}\cos\psi\right)^2 = \text{constant}.$$

Neglecting terms in $(r'/a)^2$, and denoting the mean value of the vorticity by Ω, this gives

Digitized by Google

$$\omega = \Omega\left(1 - \frac{2r'}{a}\cos\psi\right).$$

Thus, our fundamental hypothesis that ω and e are constant when logically carried out requires us to neglect the second term in (15), *i.e.*, velocities of order m/a. This consequently would lead us to neglect the second term in (14) also. This may explain the slight divergence in the results obtained by Prof. J. J. Thomson,* Mr T. C. Lewis, † and Mr W. M. Hicks. ‡ The two former obtain the result (14), but the latter differs in the value of the small term.

We have next to consider the velocity due to variation in the density ρ of the fluid. From the equations (6) and (10) of Chap. VI. of Lamb's Treatise, it follows that the velocity due to change of density is expressed by the same formula as the force due to a gravitating mass of density $-\frac{1}{4\pi\rho}\frac{\delta\rho}{\delta t}$. If the ring be of small cross section σ we may in calculating the velocity, to the same degree of accuracy as when treating the vorticity, regard it as equal to the force due to a mass of line density $-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}$ concentrated in the circular axis $r = a$ of the ring. If ρ varied with the distance from the circular axis, but was independent of ψ, the accuracy would not be seriously affected.

If w_1 and u_1 denote the component velocities parallel and perpendicular to the axis of the ring, due to the variation in density alone, then from the above remarks it follows that

$$w_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}az\int_0^\pi \frac{d\theta}{(z^2 + r^2 + a^2 - 2ra\cos\theta)^{\frac{3}{2}}} \quad \ldots \quad \ldots \quad (16),$$

$$u_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}a\int_0^\pi \frac{(r - a\cos\theta)d\theta}{(z^2 + r^2 + a^2 - 2ra\cos\theta)^{\frac{3}{2}}} \quad \ldots \quad \ldots \quad (17).$$

Thence, k having its previous meaning, we find

$$w_1 = \frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}a\frac{d}{dz}\left[\{z^2 + (r+a)^2\}^{-\frac{1}{2}}F_1(k)\right] \quad \ldots \quad \ldots \quad (18).$$

To the same degree of approximation as in the case of vorticity, we

---

* "Motion of Vortex Rings"—Equation (41), p. 33.

† "Quarterly Journal of Mathematics," XVI., 1879, pp. 338-347.

‡ "Philosophical Transactions," 1884, Part I., and 1885, Part II.

Digitized by Google

have for the velocity in the fluid comparatively near the surface of the filament

$$w_1 = -\frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}az\{z^2+(r+a)^2\}^{-\frac{3}{2}}\left[\log\left\{4\left(\frac{z^2+(r+a)^2}{z^2+(r-a)^2}\right)^{\frac{1}{2}}\right\} + \frac{4ra}{z^2+(r-a)^2}\right] \quad \ldots \quad \ldots \quad (19).$$

For points just outside the ring, using the previous notation and retaining the principal terms, we find

$$w_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\sin\psi\left(1-\frac{e}{2a}\cos\psi\right) \quad \ldots \quad \ldots \quad (20).$$

The most important term neglected is $-\frac{\sigma}{8\pi\rho}\frac{\delta\rho}{\delta t}\frac{e}{a^2}\sin\psi\log\frac{8a}{e}$.

From (17) we find

$$u_1 = \frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}a\frac{d}{dr}\left[\{z^2+(r+a)^2\}^{-\frac{1}{2}}F_1(k)\right] \quad \ldots \quad \ldots \quad (21);$$

and thence for points comparatively near the ring

$$u_1 = -\frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}a\{z^2+(r+a)^2\}^{-\frac{3}{2}}\left[(r+a)\log\left\{4\left(\frac{z^2+(r+a)^2}{z^2+(r-a)^2}\right)^{\frac{1}{2}}\right\} + 2a\frac{(r^2-a^2-z^2)}{z^2+(r-a)^2}\right] \quad \ldots \quad \ldots \quad (22).$$

For points just outside the ring we thence obtain the approximate value

$$u_1 = -\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\left[\frac{1}{a}\left(\log\frac{8a}{e}-1\right)+\frac{2}{e}\cos\psi\left(1-\frac{e}{2a}\cos\psi\right)\right] \quad \ldots \quad \ldots \quad (23).$$

The most important term neglected is $\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{e}{a^2}\cos\psi\log\frac{8a}{e}$.

From (20) and (23) we see that the velocity in the fluid just outside the filament is composed of the two components

$$u_1 = -\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}\left\{\log\frac{8a}{e}-1\right\} \quad \ldots \quad \ldots \quad (24),$$

perpendicular to the axis oz, *i.e.*, tending to increase the radius a of the circular axis, and

$$v_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\left(1-\frac{e}{2a}\cos\psi\right) \quad \ldots \quad \ldots \quad (25),$$

normal to the surface of the filament, *i.e.*, tending to increase e.

The first component u_1 represents a motion of the ring and surrounding fluid *en masse*; the second v_1 gives the rate of increase in the radius of the cross section consequent on the change in density. The slight variation in the rate of increase of e in different directions

Digitized by Google

is a phenomenon exactly similar to that illustrated by the existence of the small term in (15). A consideration of the equation of continuity shows us, precisely as in the parallel case in vorticity, that if the density be uniform over the cross section it cannot vary so that the cross section remain truly circular, unless velocities of the order $\frac{\sigma}{\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}$ be negligible. In this case the second terms in both (24) and (25) must be neglected. It would even seem at first sight that the equation of continuity was inconsistent with the existence of the principal term of (24). For, since the mass of the ring is constant, $e^2a\rho$ must be constant, and so

$$\frac{1}{\rho}\frac{\delta\rho}{\delta t}+\frac{2}{e}\frac{\delta e}{\delta t}=-\frac{1}{a}\frac{\delta a}{\delta t} \quad \ldots \quad \ldots \qquad (26).$$

But $v_1=\frac{\delta e}{\delta t}$, and so, retaining only the principal term of (25) and putting $\sigma=\pi e^2$, we get $\frac{1}{\rho}\frac{\delta\rho}{\delta t}+\frac{2}{e}\frac{\delta e}{\delta t}=0 \quad \ldots \quad \ldots \qquad (27).$

Thus it might be thought from (26) that $\frac{\delta a}{\delta t}$ must vanish.

The true explanation is that $\frac{\delta a}{\delta t}$ does not vanish, but $\frac{1}{a}\frac{\delta a}{\delta t}$ is of an order of small quantities we agreed to neglect when we came to the conclusion that the second terms in (24) and (25) were negligible. To this degree of approximation, then, we see from (27) that $\sigma\rho$ is constant, and we may replace $-\frac{\sigma}{\rho}\frac{\delta\rho}{\delta t}$ by $\frac{\delta\sigma}{\delta t}$.

Combining the effects of vorticity and change of density, and retaining only the terms consistent with an exactly circular cross section, we finally obtain for the velocities of a thin filament

$$\left.\begin{aligned} w&=\frac{m}{2\pi a}\log\frac{8a}{e}\\ u&=\frac{1}{4\pi a}\frac{\delta\sigma}{\delta t}\log\frac{8a}{e}\end{aligned}\right\} \quad \ldots \quad \ldots \qquad (28).$$

The circular axis of the ring moves on the surface formed by the revolution about oz of the curve whose differential equation is

$$\frac{dz}{dx}=\frac{w}{u}=2m/\frac{\delta\sigma}{\delta t} \quad \ldots \quad \ldots \qquad (29).$$

If the rate of increase of the cross section be uniform this forms part

Digitized by Google

of a right circular cone whose vertical angle is $2\tan^{-1}\left(\frac{1}{2m}\frac{\delta\sigma}{\delta t}\right)$.

It is easily seen that the case of a ring vortex in presence of an infinite plane, whether inclined or not to the plane of the circular axis, can be treated by the introduction of an "image" ring on the other side of the plane. The position and direction of rotation in the image were indicated in my previous paper; the cross section and density must be the same as at corresponding points in the real ring. However, the ring will remain circular with its circular axis in one plane only when it is parallel to the infinite plane, and the formulæ obtained above will be most usefully employed when the distance c of the ring from the plane is small compared to the radius a, though large compared to e. This case we proceed to treat.

Let us take the origin where the infinite plane is intersected by the common axis of the rings.

The components of the velocity at any point due to the real ring may be got from the preceding formulæ by writing $z-c$ for z, while the components of the velocity due to the image ring require the substitution of $z+c$ for z, and $-m$ for m.

For the velocity in the fluid immediately surrounding the ring the effect of the ring itself is given by (8), (13), (20), and (23), while the effect of the image may be got from (7), (12), (19), and (22) by writing $-m$ for m, $a+e\cos\psi$ for r and $2c+e\sin\psi$ for z. Combining the effects and retaining only the principal terms in accordance with the remarks already made as to the probable degree of accuracy of the method, I find

$$w=-\frac{m\cos\psi}{\pi e}-\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\sin\psi+\frac{m}{2\pi a}\log\frac{2c}{e}-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{c} \quad \ldots\ \ldots \quad (30),$$

$$u=\frac{m\sin\psi}{\pi e}-\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\cos\psi-\frac{m}{2\pi c}-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}\log\left(\frac{32a^2}{ce}\right) \quad \ldots \quad (31).$$

The two first terms in both (30) and (31) represent the velocity of circulation and the rate of increase of the radius of the cross section, while the two last terms in each represent a motion *en masse* shared by the ring and the surrounding fluid. Thus the velocity of the ring in the direction of the normal drawn *from* the infinite plane is

$$w=\frac{m}{2\pi a}\log\frac{2c}{e}-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{c} \quad \ldots\ \ldots \quad (32),$$

while the rate of increase in the radius of the ring is

Digitized by Google

$$u = \frac{\delta a}{\delta t} = -\frac{m}{2\pi c} - \frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}\log\left(\frac{32a^2}{ce}\right) \quad \ldots \quad \ldots \qquad (33).$$

Neglecting at first any change in ρ we have from (32) and (33)

$$\frac{\delta c}{\delta t} = \frac{m}{2\pi a}\log\frac{2c}{e} \quad \ldots \quad \ldots \qquad (34),$$

$$\frac{\delta a}{\delta t} = -\frac{m}{2\pi c} \quad \ldots \quad \ldots \qquad (35);$$

while $ae^2 =$ constant.

It follows that $\frac{\delta^2 c}{\delta t^2} = \frac{m^2}{4\pi^2 a^2 c}\left\{\log\left(\frac{2c}{e}\right)^2 - \frac{1}{2}\right\}$.

Thus $\frac{\delta^2 c}{\delta t^2}$ is always positive, for c must be greater than e and so $\log\left(\frac{2c}{e}\right)^2$ greater than $\frac{1}{2}$. Hence if m be negative, or the vortex be approaching the plane, its rate of approach continually diminishes; while if m be positive, or the vortex be receding, its rate of retreat continually increases so long as (34) and (35) apply.

From (35) we see that the aperture, $2a$, of the vortex increases or diminishes continually according as it is approaching to or receding from the plane.

Considering next the effect of the variation in density alone, we see from (32) and (33) that if its density be increasing the ring approaches the plane with a continually diminishing aperture, while if the density be diminishing the ring recedes from the plane with a continually increasing aperture. The exact relation between the variations in a, e and ρ is given by (26), but to the degree of accuracy obtained here this may, as in the case of a solitary ring, be replaced by (27). Using (27) in (32), and replacing σ by πe^2, we find

$$\frac{\delta c}{\delta t} = \frac{e}{2c}\frac{\delta e}{\delta t} = \frac{1}{4\pi c}\frac{\delta\sigma}{\delta t};$$

whence it follows that $c^2 - \frac{1}{2}e^2 = c^2 - \frac{\sigma}{2\pi} =$ constant $\quad \ldots \quad \ldots \qquad (36).$

In the cases to which our formulæ can be satisfactorily applied e/c is small, and so the total increase or diminution in the distance of the ring from the plane due to change in density alone is also small. Thus, in general, the effects of the vorticity will be much more important than the effects of variation in density. We conclude that if a vortex ring approach an infinite plane, its rate of approach is slightly greater if its density be increasing, and slightly less if its

Digitized by Google

density be diminishing, than it would be if the density remained constant.

The most important terms due to the action of the image ring which we have neglected in (30) and (31) respectively may without much difficulty be found to be

$$\left.\begin{aligned} w_2 &= A\cos(\psi - a) \\ u_2 &= A\sin(\psi - a) \end{aligned}\right\} \quad \dots \quad \dots \quad (37);$$

where, for shortness,

$$\left.\begin{aligned} A &= \frac{e}{4\pi c^2}\left\{m^2 + \left(\frac{\sigma}{2\rho}\frac{\delta\rho}{\delta t}\right)^2\right\}^{\frac{1}{2}} \\ a &= \tan^{-1}\left(\frac{\sigma}{2m\rho}\frac{\delta\rho}{\delta t}\right) \end{aligned}\right\} \quad \dots \quad \dots \quad (38).$$

Since, as already explained, we are not warranted in retaining velocities of order $\frac{m}{a}$ it follows that the above terms represent an appreciable effect only when the vortex approaches so close to the plane that ae/c^2 becomes large.

From (37) it follows that the cross section of the filament tends to become slightly elliptical, the axes of the ellipse making with the infinite plane the angles $\frac{a}{2} \pm \frac{\pi}{4}$. These axes are thus equally inclined to the plane when the fluid is incompressible. When the deviation of the cross section from the circular form becomes appreciable the accuracy of the preceding formulæ will be lessened, and they can certainly not be applied to the case of a vortex whose distance from an infinite plane is of the same order of quantities as the diameter of its cross section.

If in (32) – (36) we write $c/2$ for c, we get the case of two precisely equal ring vortices, with vorticities, however, in opposite directions, at a distance c. If we suppose a to become infinite, we deduce formulæ applicable to the case of straight vortex filaments. In particular, it will be noticed that (36) leads at once to a special case of the formula obtained in my previous paper for the distance of two straight vortices.

Digitized by Google

Similitude and Inversion.

By J. S. Mackay, LL.D.

The following paper contains little that can be regarded as new mathematical information. It aims only at showing, or rather at emphasising, the correspondence which exists between two geometrical theories which are related to each other in the same way as the arithmetical theories of multiplication and division. Such value, therefore, as it possesses is primarily pedagogical.

Attention should perhaps be drawn to the (unusual) use of the word "similar" in the sense of "similar and similarly situated," or "homothetic." The word homothesis (French geometers have adopted a curious form *homothétie*) is not naturalised in English; otherwise homothesis and homothetic might have been used instead of similitude and similar. The pair, similitude and homothetic, are somewhat incongruous.

The term antiparallel (said to have been first employed with a definite geometrical meaning by Antoine Arnauld in 1667) was not uncommon in English mathematical writings about a century ago. It is again, and deservedly, coming into use, and the following definition of it may be given:

Two straight lines intersecting the sides of an angle, or its vertically opposite angle, are antiparallel when the first straight line makes with one of the sides of the given angle an angle equal to that which the second straight line makes with the other side.

Figure 37′.

Thus, if $\angle OM'P' = \angle OPM$, then M′P′ is antiparallel to MP with respect to $\angle O$.

The following are some properties connected with antiparallels, the proof of which need not be given here.

(1) If M′P′ is antiparallel to MP with respect to $\angle O$, then MM′ is antiparallel to PP′.

(2) The four points M, M′, P, P′ are concyclic.

(3) The rectangles OM·OM′, OP·OP′ are equal.

(4) The triangles OMP, OM′P′ are similar.

Digitized by Google

Similitude.

§ 1. *Definition.*—If three collinear points O, P, P′ be given, any two of them may be considered similar to each other, and the third may be taken as their centre of similitude. The ratio of the distances of the third point from the other two is called the ratio of similitude.

Thus, if O be chosen as centre of similitude, and P′ be considered similar to P, the ratio of similitude is OP : OP′.

When P and P′ are on the same side of O, the ratio of similitude is positive, since OP and OP′ are drawn in the same direction; when P and P′ are on opposite sides of O, the ratio of similitude is negative, since OP and OP′ are drawn in opposite directions.

When a given point P is variable, that is, when it occupies a series of consecutive positions, the point P′ similar to it will also occupy a series of consecutive positions; in other words, when a given point P describes a certain curve, the similar point P′ will describe the similar curve.

§ 2. Given a centre of similitude O, and a ratio of similitude $r : r'$, to find the point similar to a given point P.

Figure 37.

Through O draw any straight line OM, and make $OM = r$, $OM' = r'$. M and M′ will be on the same side of O or on opposite sides of O, according as $r : r'$ is positive or negative. Join MP, OP, and through M′ draw M′P′ parallel to MP, and meeting OP in P′.

Then P′ is similar to P.

For $OP : OP' = OM : OM' = r : r'$.

§ 3. Given two points P, P′ similar to each other, and a ratio of similitude $r : r'$, to find the centre of similitude.

The centre of similitude is determined by joining PP′ and dividing it externally or internally at O so that the segments OP, OP′ may have the given ratio.

§ 4. Given two pairs of similar points P and P′, Q and Q′, to find the centre of similitude.

Case 1. When the four points are not collinear.

Digitized by Google

Inversion.

§ 1′. *Definition.*—If three collinear points O, P, P′ be given, any two of them may be considered inverse to each other, and the third may be taken as their centre of inversion. The rectangle under the distances of the third point from the other two is called the rectangle of inversion.

Thus, if O be chosen as centre of inversion, and P′ be considered inverse to P, the rectangle of inversion is OP·OP′.

When P and P′ are on the same side of O, the rectangle of inversion is positive, since OP and OP′ are drawn in the same direction; when P and P′ are on opposite sides of O, the rectangle of inversion is negative, since OP and OP′ are drawn in opposite directions.

When a given point P is variable, that is, when it occupies a series of consecutive positions, the point P′ inverse to it will also occupy a series of consecutive positions; in other words, when a given point P describes a certain curve, the inverse point P′ will describe the inverse curve.

§ 2′. Given a centre of inversion O, and a rectangle of inversion $r \cdot r'$ to find the point inverse to a given point P.

Figure 37′.

Through O draw any straight line OM, and make $OM = r$, $OM' = r'$. M and M′ will be on the same side of O or on opposite sides of O, according as $r \cdot r'$ is positive or negative. Join MP, OP, and through M′ draw M′P′ antiparallel to MP with respect to angle MOP, and meeting OP in P′.

Then P′ is inverse to P.

For $OP \cdot OP' = OM \cdot OM' = r \cdot r'$.

§ 3′. Given two points P, P′ inverse to each other, and a rectangle of inversion $r \cdot r'$, to find the centre of inversion.

The centre of inversion is determined by joining PP′ and dividing it externally or internally at O so that the segments OP, OP′ may contain the given rectangle.

§ 4′. Given two pairs of inverse points P and P′, Q and Q′, to find the centre of inversion.

Case 1. When the four points are not collinear.

Digitized by Google

Join PP′, QQ′, and let them intersect at O. Then O is the centre of similitude.

This follows from § 1.

Case 2. When the four points are collinear.

FIRST METHOD.

Figure 38.

Take any point M not collinear with the four points, and join PM, QM.

Through P′ draw a straight line parallel to PM; through Q′ draw a straight line parallel to QM; and let these straight lines intersect at M′.

MM′ will intersect PP′ at O, the centre of similitude.

For $$OP : OP' = OM : OM',$$
$$= OQ : OQ'.$$

SECOND METHOD.

Figure 39.

Take any point M not collinear with the four points, and join PM, QM.

Through Q′ describe a circle passing through P, M; through P′ describe a circle passing through Q, M; and let these circles intersect at M′.

MM′ will intersect PP′ at O, the centre of similitude.

For $$OP \cdot OQ' = OM \cdot OM'.$$
$$= OQ \cdot OP';$$

therefore $$OP : OP' = OQ : OQ'.$$

§ 5. It will be seen from the two preceding methods of solution that, when two pairs of similar points happen to be collinear, two pairs of inverse points also are obtained.

For the equality of the ratios OP : OP′, OQ : OQ′ necessitates the equality of the rectangles OP·OQ′, OQ·OP′.

§ 6. Given a centre of similitude O, and a ratio of similitude $r : r'$, to find the system of points similar to a given system A, B, C, ...

Figure 41.

Join O to A, B, C, ...; and on OA, OB, OC, ... find A′, B′, C′, ... such that

$$OA : OA' = OB : OB' = OC : OC' = \ldots = r : r'.$$

Digitized by Google

Join PP′, QQ′, and let them intersect at O. Then O is the centre of inversion.

This follows from § 1′.

Case 2. When the four points are collinear.

FIRST METHOD.

Figure 38′.

Take any point M not collinear with the four points, and join PM, QM.

Through P′ describe a circle passing through P, M; through Q′ describe a circle passing through Q, M; and let these circles intersect at M′.

MM′ will intersect PP′ at O, the centre of inversion.

For $$OP \cdot OP' = OM \cdot OM',$$
$$= OQ \cdot OQ'.$$

SECOND METHOD.

Figure 39′.

Take any point M not collinear with the four points, and join PM, QM.

Through Q′ draw a straight line parallel to PM; through P′ draw a straight line parallel to QM; and let these straight lines intersect at M′.

MM′ will intersect PP′ at O, the centre of inversion.

For $$OP : OQ' = OM : OM',$$
$$= OQ : OP';$$

therefore $$OP \cdot OP' = OQ \cdot OQ'.$$

§ 5′. It will be seen from the two preceding methods of solution that, when two pairs of inverse points happen to be collinear, two pairs of similar points also are obtained.

For the equality of the rectangles OP·OP′, OQ·OQ′ necessitates the equality of the ratios OP : OQ′, OQ : OP′.

§ 6′. Given a centre of inversion O, and a rectangle of inversion $r \cdot r'$, to find the system of points inverse to a given system A, B, C, ...

Figure 41′.

Join O to A, B, C, ...; and on OA, OB, OC, ... find A′, B′, C′, ... such that

$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \ldots = r \cdot r'.$$

Digitized by Google

§ 7. If two systems of points be similar, the straight line joining any pair of points in the one is parallel to the straight line joining the corresponding pair in the other.

This follows from § 2.

§ 8. If two systems of points be similar, and three points of the first system be collinear, the three corresponding points of the second system will also be collinear.

Figure 40.

Let the system A, B, C be similar to the system A′, B′, C′, and let A, B, C be collinear.

Since, by § 7, A′B′ is parallel to AB,

therefore $\angle OB'A' = \angle OBA.$

Similarly $\angle OB'C' = \angle OBC;$

therefore $\angle OB'A' + \angle OB'C' = \angle OBA + \angle OBC,$

$= 2$ right angles;

therefore A′, B′, C′ are collinear.

Hence the curve similar to a straight line is a straight line.

§ 9. If two systems of points be similar, with respect to a centre of similitude O and a ratio of similitude $r : r'$, then for every two points A, B and the points A′, B′ similar to them

$$A'B' : AB = r' : r.$$

Figure 41.

Since A′B′ is parallel to AB,

therefore triangles OA′B′, OAB are equiangular;

therefore $A'B' : AB = OA' : OA,$

$= r' : r.$

If p', p be the perpendiculars from O on A′B′ and AB, it also follows from the equiangularity of the triangles OA′B′, OAB that

$$A'B' : AB = p' : p,$$

§ 10. If two systems of points be similar, then for every three points A, B, C and the points A′, B′, C′ similar to them

$$B'C' : C'A' : A'B' = BC : CA : AB.$$

Digitized by Google

§ 7′. If two systems of points be inverse, the straight line joining any pair of points in the one is antiparallel to the straight line joining the corresponding pair in the other.

This follows from § 2′.

§ 8′. If two systems of points be inverse, and three points of the first system be collinear, the three corresponding points of the second system will not in general be collinear.

Figure 40′.

Let the system A, B, C be inverse to the system A′, B′, C′, and let A, B, C be collinear.

Since, by § 7′, A′B′ is antiparallel to AB,

therefore $\angle OA'B' = \angle OBA.$

Similarly $\angle OC'B' = \angle OBC$

therefore $\angle OA'B' + \angle OC'B' = \angle OBA + \angle OBC,$

$= 2$ right angles;

which is impossible, if A′, B′, C′ be collinear.

Hence the curve inverse to a straight line is not in general a straight line.

§ 9′. If two systems of points be inverse, with respect to a centre of inversion O and a rectangle of inversion $r \cdot r'$, then for every two points A, B and the points A′, B′ inverse to them

$$A'B' : AB = r \cdot r' : OA \cdot OB.$$

Figure 41′.

Since A′B′ is antiparallel to AB with respect to angle AOB, therefore triangles OA′B′, OAB are equiangular;

therefore $A'B' : AB = OA' : OB = OA \cdot OA' : OA \cdot OB,$

$= r \cdot r' : OA \cdot OB.$

If p', p be the perpendiculars from O on A′B′ and AB, it also follows from the equiangularity of the triangles OA′B′, OAB that

$$A'B' : AB = p' : p.$$

§ 10′. If two systems of points be inverse, then for every three points A, B, C and the points A′, B′, C′ inverse to them

$$B'C' : C'A' : A'B' = OA \cdot BC : OB \cdot CA : OC \cdot AB.$$

Digitized by Google

Figure 41.

For $$B'C' = BC.\frac{r'}{r},\ C'A' = CA.\frac{r'}{r};$$

therefore $$\frac{B'C'}{C'A'} = BC.\frac{r'}{r}/CA.\frac{r'}{r},$$
$$= BC/CA.$$

§ 11. If two systems of points be similar, then for every four points A, B, C, D and the points A′, B′, C′, D′ similar to them

$$B'C'\cdot A'D' : C'A'\cdot B'D' : A'B'\cdot C'D' =$$
$$BC\cdot AD : CA\cdot BD : AB\cdot CD.$$

Figure 41.

For $$B'C'\cdot A'D' = BC.\frac{r'}{r}.AD.\frac{r'}{r},$$

and $$C'A'\cdot B'D' = CA.\frac{r'}{r}.BD.\frac{r'}{r};$$

therefore $B'C'\cdot A'D' : C'A'\cdot B'D' = BC\cdot AD : CA\cdot BD.$

§ 12. Every straight line passing through a centre of similitude cuts two similar curves C, C′ at similar points.

This follows from § 1.

§ 13. Every straight line passing through a centre of similitude and touching a curve C will also touch the similar curve C′.

Figure 42.

Let a straight line through the centre of similitude O cut C at P and Q, then it will cut C′ at P′ and Q′ the points similar to P and Q. Now, when the points P and Q move up to each other and ultimately coincide, that is, when the straight line touches C, the points P′ and Q′ will move up to each other and ultimately coincide, that is, the straight line will touch C′.

§ 14. If on two similar curves C, C′ similar points P, P′ be taken, the tangents at P, P′ make equal angles with OPP′.

Figure 43.

On C take any point Q near to P, and on C′ find the point Q′ similar to Q. Draw the secants QPR, Q′P′R′.

Digitized by Google

Figure 41′.

For $$B'C' = BC.\frac{rr'}{OB\cdot OC},\quad C'A' = CA.\frac{r.r'}{OC\cdot OA};$$

therefore $$\frac{B'C'}{C'A'} = BC.\frac{rr'}{OB\cdot OC}/CA.\frac{rr'}{OC\cdot OA},$$

$$= OA\cdot BC/OB\cdot CA.$$

§ 11′. If two systems of points be inverse, then for every four points A, B, C, D and the points A′, B′, C′, D′ inverse to them

$$B'C'\cdot A'D' : C'A'\cdot B'D' : A'B'\cdot C'D' =$$
$$BC\cdot AD : CA\cdot BD : AB\cdot CD.$$

Figure 41′.

For $$B'C'\cdot A'D' = BC.\frac{rr'}{OB\cdot OC}.AD.\frac{rr'}{OA\cdot OD},$$

and $$C'A'\cdot B'D' = CA.\frac{rr'}{OC\cdot OA}.BD.\frac{rr'}{OB\cdot OD};$$

therefore $$B'C'\cdot A'D' : C'A'\cdot B'D' = BC\cdot AD : CA\cdot BD.$$

§ 12′. Every straight line passing through a centre of inversion cuts two inverse curves C, C′ at inverse points.

This follows from § 1′.

§ 13′. Every straight line passing through a centre of inversion and touching a curve C will also touch the inverse curve C′.

Figure 42′.

Let a straight line through the centre of inversion O cut C at P and Q, then it will cut C′ at P′ and Q′ the points inverse to P and Q. Now, when the points P and Q move up to each other and ultimately coincide, that is, when the straight line touches C, the points P′ and Q′ will move up to each other and ultimately coincide, that is, the straight line will touch C′.

§ 14′. If on two inverse curves C, C′ inverse points P, P′ be taken, the tangents at P, P′ make supplementary angles with OPP′.

Figure 43′.

On C take any point Q near to P, and on C′ find the point Q′ inverse to Q. Draw the secants QPR, Q′P′R′.

Digitized by Google

Then the secants QR, Q'R' are parallel;

therefore $\angle OQP = \angle OQ'P'$.

Now, when Q moves to coincidence with P, that is, when the secant QR becomes the tangent PT, Q' moves to coincidence with P', that is, the secant Q'R' becomes the tangent P'T'.

Also when Q moves to coincidence with P, $\angle OQP$ becomes $\angle OPS$; and when Q' moves to coincidence with P', $\angle OQ'P'$ becomes $\angle OP'S'$;

therefore $\angle OPS = \angle OP'S'$;

therefore $\angle OPT = \angle OP'T'$.

§ 15. If two curves intersect each other at any angle, the curves similar to them intersect each other at the same angle.

Figure 44.

Let the two curves C and D intersect each other at P; then C' and D' the curves similar to them will intersect each other at P' the point similar to P.

Draw PT, PU tangents to C and D; and P'T', P'U' tangents to C' and D'; and let O be the centre of similitude.

Then $\angle OPT = \angle OP'T'$

and $\angle OPU = \angle OP'U'$;

therefore $\angle OPT - \angle OPU = \angle OP'T' - \angle OP'U'$;

therefore $\angle TPU = \angle T'P'U'$.

Hence, if two curves touch each other at any point P, the curves similar to them will touch each other at the similar point P'.

§ 16. If two curves C', C'' be both similar to a curve C with respect to the same centre of similitude O, then C', C'' are similar to each other.

Let $r : r'$ and $r : r''$ be the two ratios of similitude.

Take any point P on C, and the similar points P', P'' on C' and C''.

Then $OP : OP' = r : r'$,

$OP : OP'' = r : r''$;

therefore $\frac{OP'}{OP} \Big/ \frac{OP''}{OP} = \frac{r'}{r} \Big/ \frac{r''}{r}$;

therefore $OP' : OP'' = \frac{r'}{r} : \frac{r''}{r}$

$= $ a constant ratio.

Hence C' is similar to C''.

Digitized by Google

Then the secants QR, Q'R' are antiparallel;

therefore $\angle OPQ = \angle OQ'P'$.

Now, when Q moves to coincidence with P, that is, when the secant QR becomes the tangent PT, Q' moves to coincidence with P', that is, the secant Q'R' becomes the tangent P'T'.

Also when Q moves to coincidence with P, $\angle OPQ$ becomes $\angle OPT$; and when Q' moves to coincidence with P', $\angle OQ'P'$ becomes $\angle OP'S'$;

therefore $\angle OPT = \angle OP'S'$;

therefore $\angle OPT =$ supplement of $\angle OP'T'$.

§ 15'. If two curves intersect each other at any angle, the curves inverse to them intersect each other at the same angle.

Figure 44'.

Let the two curves C and D intersect each other at P; then C' and D' the curves inverse to them will intersect each other at P' the point inverse to P.

Draw PT, PU tangents to C and D; and P'T', P'U' tangents to C' and D'; and let O be the centre of inversion.

Then $\angle OPT =$ supplement of $\angle OP'T'$

and $\angle OPU =$ supplement of $\angle OP'U'$;

therefore $\angle OPT - \angle OPU =$ supplement of $\angle OP'T' -$ supplement of $\angle OP'U'$;

therefore $\angle TPU = \angle T'P'U'$.

Hence, if two curves touch each other at any point P, the curves inverse to them will touch each other at the inverse point P'.

§ 16'. If two curves C', C'' be both inverse to a curve C with respect to the same centre of inversion O, then C', C'' are similar to each other.

Let $r.r'$ and $r.r''$ be the two rectangles of inversion.

Take any point P on C, and the inverse points P', P'' on C' and C''.

Then $OP \cdot OP' = r.r'$,

$OP \cdot OP'' = r.r''$;

therefore $OP \cdot OP' / OP \cdot OP'' = r.r' / r.r''$;

therefore $OP' : OP'' = r.r' : r.r''$

$=$ a constant ratio.

Hence C' is similar to C''.

Digitized by Google

§ 17. Given a centre of similitude O and a ratio of similitude $r : r'$, to find the curve similar to a given straight line PQ.

Case 1. When PQ passes through O.

Figure 45.

Take any point P in PQ, and find in OP the point P′ such that $OP : OP' = r : r'$. P and P′ will be on the same side of O or on opposite sides of O according as $r : r'$ is positive or negative.

Since the ratio OP : OP′ is fixed, as OP increases OP′ will also increase; and as OP diminishes OP′ will also diminish. In other words, as P moves farther and farther from O, P′ will also move farther and farther from O; as P moves nearer and nearer to O, P′ will also move nearer and nearer to O. And consequently when P is infinitely distant from O, P′ will also be infinitely distant from O; when P coincides with O, P′ will also coincide with O. Hence when P describes from right to left or from left to right the straight line PQ, P′ will also describe from right to left or from left to right the same straight line.

Case 2. When PQ does not pass through O.

Figure 46.

Through O draw OP perpendicular to PQ, and find in OP the point P′ such that $OP : OP' = r : r'$; through P′ draw P′Q′ perpendicular to OP′.

This perpendicular is the curve similar to PQ.

Take any point Q in PQ, join OQ, and let it meet the perpendicular in Q′.

Since angles OPQ, OP′Q′ are right,

therefore PQ and P′Q′ are parallel;

therefore $$OQ : OQ' = OP : OP' = r : r'.$$

Hence Q′ is the point similar to Q, and as Q was any point whatever in PQ, therefore all the points in PQ have the points similar to them situated in P′Q′; that is, the straight line P′Q′ is the curve similar to the given straight line PQ.

§ 18. If the straight line P′Q′ is similar to the straight line PQ with respect to a given centre of similitude O, the reciprocal relation also holds good, namely, that the straight line PQ is similar to the straight line P′Q′ with respect to the same centre of similitude.

Digitized by Google

§ 17′. Given a centre of inversion O and a rectangle of inversion $r.r'$, to find the curve inverse to a given straight line PQ.

Case 1. When PQ passes through O.

Figure 45′.

Take any point P in PQ, and find in OP the point P′ such that $OP \cdot OP' = r.r'$. P and P′ will be on the same side of O or on opposite sides of O according as $r.r'$ is positive or negative.

Since the rectangle $OP \cdot OP'$ is fixed, as OP increases OP′ will diminish; and as OP diminishes OP′ will increase. In other words, as P moves farther and farther from O, P′ will move nearer and nearer to O; as P moves nearer and nearer to O, P′ will move farther and farther from O. And consequently when P is infinitely distant from O, P′ will coincide with O; when P coincides with O, P′ will be infinitely distant from O. Hence when P describes from right to left or from left to right the straight line PQ, P′ will also describe from left to right or from right to left the same straight line.

Case 2. When PQ does not pass through O.

Figure 46′.

Through O draw OP perpendicular to PQ, and find in OP the point P′ such that $OP \cdot OP' = r.r'$; on OP′ as diameter describe a circle.

This circle is the curve inverse to PQ.

Take any point Q in PQ, join OQ, and let it meet the circle in Q′; and join P′Q′.

Since angles OPQ, OQ′P′ are right,

therefore PQ and P′Q′ are antiparallel with respect to angle POQ;

therefore $OQ \cdot OQ' = OP \cdot OP' = r.r'$.

Hence Q′ is the point inverse to Q, and as Q was any point whatever in PQ, therefore all the points in PQ have the points inverse to them situated on the circle OP′Q′; that is, the circle OP′Q′ is the curve inverse to the given straight line PQ.

§ 18′. If the circle OP′Q′ is inverse to the straight line PQ with respect to a given centre of inversion O, the reciprocal relation also holds good, namely, that the straight line PQ is inverse to the circle OP′Q′ with respect to the same centre of inversion.

Digitized by Google

§ 19. Given a centre of similitude O, and a ratio of similitude $r : r'$, to find the curve similar to a circle.

Case 1. When the circle passes through O.

Figure 47.

Let OPQ be the given circle.

Through O draw the diameter OP, and in OP find the point P′ such that $OP : OP' = r : r'$; on OP′ as diameter describe the circle OP′Q′.

This circle is the curve similar to OPQ.

Take any point Q in OPQ, join OQ, and let it meet the circle OP′Q′ at Q′. Join PQ, P′Q′.

Since angles OQP and OQ′P′ are right,

therefore PQ and P′Q′ are parallel;

therefore $OQ : OQ' = OP : OP' = r : r'$.

Hence Q′ is the point similar to Q, and as Q was any point whatever in OPQ, therefore all the points in OPQ have the points similar to them situated in OP′Q′; that is, the circle OP′Q′ is the curve similar to the given circle OPQ.

Case 2. When the circle does not pass through O.

Figure 48.

Let C be the centre of the given circle.

Take any point P on the circle C, join OP, and in OP find P′ similar to P. Join PC, and at P′ make angle OP′C′ equal to angle OPC, and let P′C′ meet OC at C′. With C′ as centre and C′P′ as radius describe a circle.

This circle will be similar to the circle C.

Since $\angle OPC = \angle OP'C'$,

therefore PC and P′C′ are parallel;

therefore $OP : OP' = CP : C'P'$.

Hence
$$C'P' = CP.\frac{OP'}{OP}$$
$$= \text{a constant},$$

since CP is a constant length, and $OP : OP'$ is a constant ratio.

Digitized by Google

§ 19′. Given a centre of inversion O, and a rectangle of inversion $r \cdot r'$, to find the curve inverse to a circle.

Case 1. When the circle passes through O.

Figure 47′.

Let OPQ be the given circle.

Through O draw the diameter OP, and in OP find the point P′ such that $OP \cdot OP' = r \cdot r'$; at P′ draw P′Q′ perpendicular to OP′.

This perpendicular is the curve inverse to OPQ.

Take any point Q in OPQ, join OQ, and let it meet the perpendicular P′Q′ at Q′. Join PQ.

Since angles OQP and OP′Q′ are right,

therefore PQ and P′Q′ are antiparallel with respect to angle POQ;

therefore $$OQ \cdot OQ' = OP \cdot OP' = r \cdot r'.$$

Hence Q′ is the point inverse to Q, and as Q was any point whatever in OPQ, therefore all the points in OPQ have the points inverse to them situated in P′Q′; that is, the straight line P′Q′ is the curve inverse to the given circle OPQ.

Case 2. When the circle does not pass through O.

Figure 48′.

Let C be the centre of the given circle.

Take any point P on the circle C, join OP, and in OP find P′ inverse to P. Join PC, and at P′ make angle OP′C′ equal to the supplement of angle OPC, and let P′C′ meet OC at C′. With C′ as centre and C′P′ as radius describe a circle.

This circle will be inverse to the circle C.

Let OP meet the circle C again at Q, and join CQ.

Since supplement of $\angle OPC = \angle OP'C'$,

therefore $$\angle OQC = \angle OP'C';$$

therefore QC and P′C′ are parallel;

therefore $$OQ : OP' = CQ : C'P'.$$

Hence $$C'P' = CQ \cdot \frac{OP'}{OQ} = CQ \cdot \frac{OP \cdot OP'}{OP \cdot OQ}$$

$$= \text{a constant},$$

since CQ is a constant length, and $OP \cdot OP'$, $OP \cdot OQ$ are constant rectangles.

Digitized by Google

Again $$OP : OP' = OC : OC';$$

therefore $$OC' = OC.\frac{OP'}{OP}$$

$$= \text{a constant.}$$

therefore C′ is a fixed point.

Since C′ is a fixed point, and C′P′ is of constant length, therefore the locus of P′ is the circle C′.

§ 20. If the circle C′ is similar to the circle C with respect to a given centre of similitude, the reciprocal relation also holds good, namely, that C is similar to C′ with respect to the same centre of similitude.

§ 21. Given two circles C, C′ (whose radii are c, c') similar to each other, to find the centre of similitude.

Figure 48.

It will be seen from the construction and the reasoning in § 19, Case 2, that two circles have two centres of similitude, external or internal, according as their ratio of similitude is positive or negative.

Since $$OC : OC' = CP : C'P',$$

$$= c : c',$$

these centres of similitude are found by dividing CC′, the distance between the centres of the two circles, externally or internally, in the ratio of the radii.

§ 22. Given two circles C, C′ (whose radii are c, c') similar to each other, to find the ratio of similitude.

Figure 48.

Find O the external or internal centre of similitude by dividing CC′ externally or internally in the ratio $c : c'$; draw any straight line OPQ cutting the circles C, C′ in the pairs of similar points P and P′, Q and Q′; and join CP, C′P′.

Then $$c : c' = CP : C'P',$$

$$= OP : OP';$$

therefore the ratio of similitude of two circles similar to each other is the ratio of their radii.

Digitized by Google

Again $$OQ : OP' = OC : OC';$$

therefore $$OC' = OC.\frac{OP'}{OQ} = OC.\frac{OP \cdot OP'}{OP \cdot OQ}$$
$$= \text{a constant};$$

therefore C′ is a fixed point.

Since C′ is a fixed point, and C′P′ is of constant length, therefore the locus of P′ is the circle C′.

§ 20′. If the circle C′ is inverse to the circle C with respect to a given centre of inversion, the reciprocal relation also holds good, namely, that C is inverse to C′ with respect to the same centre of inversion.

§ 21′. Given two circles C, C′ (whose radii are c, c') inverse to each other, to find the centre of inversion.

Figure 48′.

It will be seen from the construction and the reasoning in § 19′, Case 2, that two circles have two centres of inversion, external or internal, according as their rectangle of inversion is positive or negative.

Since $$OC : OC' = CQ : C'P',$$
$$= c : c',$$

these centres of inversion are found by dividing CC′, the distance between the centres of the two circles, externally or internally, in the ratio of the radii.

§ 22′. Given two circles C, C′ (whose radii are c, c') inverse to each other, to find the rectangle of inversion.

Figure 48′.

Find O the external or internal centre of inversion by dividing CC′ externally or internally in the ratio $c : c'$; draw any straight line OPQ cutting the circles C, C′ in the pairs of inverse points P and P′, Q and Q′; and join CQ, C′P′.

Then $$c : c' = CQ : C'P',$$
$$= OQ : OP' = OP \cdot OQ : OP \cdot OP';$$

therefore the rectangle of inversion of two circles inverse to each other is a fourth proportional to c, c', and OP · OQ, the potency of O with respect to the circle C.

Digitized by Google

§ 23. A circle is similar to itself with respect to any centre of similitude O, when the ratio of similitude is unity.

For if P be any point on the circle,
since $OP : OP' = 1$,
therefore P′ is the same point as P.
Hence as P describes clockwise or counterclockwise the circumference of the given circle, P′ describes clockwise or counterclockwise the same circumference.

§ 24. If two circles be similar to each other their centres are similar points.

Figure 49.

Let C, C′ be two circles similar to each other, O their centre of similitude, and let OT touch the circle C at T. Then, by § 13, OT will touch the circle C′ at T′ the point similar to T. Join CT, C′T′.

Now, if the centre C be considered a point in the figure C, the point similar to it will, by § 12, be situated on OC. It will also, by § 7, be situated on a straight line through T′ parallel to TC. But since $\angle$ OTC and $\angle$ OT′C′ are right, therefore T′C′ is parallel to TC; therefore C′ is the point similar to C.

§ 25. The property of § 7 as applied to the circle may be enunciated thus:

If two circles be similar to each other the chord joining any two points of the one intersects the chord joining the two similar points of the other on the straight line at infinity.

The property of § 14 thus:

If two circles be similar to each other the tangents at two similar points of them intersect on the straight line at infinity.

Digitized by Google

§ 23′. A circle is inverse to itself with respect to any centre of inversion O, when the rectangle of inversion is the potency of O with respect to the given circle.

For if P be any point on the circle,
since $OP \cdot OP' =$ the potency of O with respect to the given circle,
therefore P′ is the point where OP cuts the given circle a second time.
Hence as P describes clockwise or counterclockwise the circumference of the given circle, P′ describes counterclockwise or clockwise the same circumference.

§ 24′. If two circles be inverse to each other their centres are not inverse points.

Figure 49′.

Let C, C′ be two circles inverse to each other, O their centre of inversion, and let OT touch the circle C at T. Then, by § 13′, OT will touch the circle C′ at T′ the point inverse to T. Join CT, C′T′.

Now, if the centre C be considered a point in the figure C, the point inverse to it will, by § 12, be situated on OC. It will also, by § 7′, be situated on a straight line through T′ antiparallel to TC. But since $\angle$ OTC and $\angle$ OT′C′ are right, therefore T′C′ is parallel to TC; therefore C′ is not the point inverse to C.

§ 25′. The property of § 7′ as applied to the circle may be enunciated thus:

If two circles be inverse to each other the chord joining any two points of the one intersects the chord joining the two inverse points of the other on the radical axis of the two circles.

The property of § 14′ thus:

If two circles be inverse to each other the tangents at two inverse points of them intersect on the radical axis of the two circles.

Digitized by Google

Eighth Meeting, June 8th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

A Construction for the Brocard Points.

By R. E. ALLARDICE, M.A.

The following note may be considered as an addendum to the paper by me on pp. 42-47 of this volume of the *Proceedings*. In that paper it is shown how to inscribe in a triangle ABC, a triangle DEF, such that the perpendiculars to the sides of ABC, drawn through the points D, E, F, shall be concurrent in a point P. This is done by constructing on each of the sides of ABC a triangle similar to DEF; then O the point of concurrence of the three lines joining the vertices of ABC to the vertices of these triangles is the point "inverse" to P. The question, then, naturally arises, What must be the shape of the triangle DEF in order that the point P may be one of the Brocard points, and, as a consequence, O the other one? and the answer is easily seen to be that DEF must be similar to ABC. Hence the following construction:—

On the sides BC, CA, make the triangles CEB, CAF, similar to ABC; then the point of concurrence of AE and BF is one of the Brocard points. The point E may be obtained by drawing BE parallel to AC and making BE a third proportional to AC and CB; and a similar construction may be given for the point F.

If, instead of making CEB and CAF similar to ABC (where the correspondence of vertices is indicated by the order in which the triangle is named), we make BCE and FCA similar to ABC, we obtain the second Brocard point.

Digitized by Google

ON THE TEACHING OF ARITHMETIC.

REPORT drawn up by the Committee appointed 13th January 1888, and approved by the Society, 8th June 1888.

---

1. The following report is not a syllabus of arithmetical teaching, far less a formal treatise, but an attempt to reduce to some order the miscellaneous suggestions at the disposal of the committee.* These suggestions are roughly classified into seven groups. Arithmetic includes both a science and an art, or a theory and applications of it to the wants of life. Of the seven divisions in this report the first four have mainly to do with the art, the next two with the science of arithmetic (the division is not observed with thoroughgoing strictness), while the seventh division contains a few general hints for teaching. These divisions may be briefly styled :—

(1.) TERMS, (2.) MEASURES and CONSTANTS, (3.) PROCESSES, (4.) COMPUTATION, (5.) THEORY, (6.) LINKS, or ideas leading to higher mathematics, and (7.) GENERAL HINTS.

I. TERMS.

2. The meaning of certain terms relating to quantity falls to be taught under Arithmetic.

(*a*) Such as occur frequently, as

Sum, difference, product, quotient, fraction, ratio, reciprocal, prime, odd and even numbers, measure, multiple, power, root, &c.

Per cent., interest, discount, stock, area, volume, average, &c.

(*b*) Such as occur less frequently, as

Index, logarithm, surd, commensurable, &c.

Specific gravity, velocity, acceleration, work, horse-power.

Of these (*a*) should be thoroughly mastered, while (*b*) should be left for advanced pupils, or explained as they occur in problems.

To facilitate the mastery of several terms in (*a*), the pupil should be made to use, instead of the general word "answer," the particular term appropriate to the case, as sum, difference, G.C.M., price per yard, &c.

---

* The Committee was as follows: Dr J. S. Mackay, Messrs A. J. G. Barclay, A. Y. Fraser (Convener), W. J. Macdonald, and Robert Robertson (who was prevented by domestic bereavement from attending the meetings). At the later meetings Mr R. E. Allardice was present by invitation of the Committee. The suggestions above referred to were made at the December and January meetings, and also by members in communications to the Secretary.

Digitized by Google

II. Measures and Constants.

3. Of the tables of measures to be committed to memory, the following are recommended as necessary and sufficient:—

Money.

12 pence (*d.*) = 1 shilling (*s.* or /)
20 shillings = 1 *pound* (*£.*)

Weight.

16 ounces (*oz.*) = 1 *pound* (*lb.*)
112 pounds = 1 hundredweight (*cwt.*)
20 hundredweights = 1 ton.

[7000 grains (*gr.*) = 1 pound. 14 pounds = 1 stone (*st.*)]

Length.

12 inches (*in.*) = 1 foot (*ft.*)
3 feet = 1 *yard* (*yd.*)
1760 yards = 1 mile (*ml.*)

[22 yards = 1 chain (*ch.*) = 100 links (*lk.*)]

Surface.

144 or 12 × 12 square inches (*sq. in.*) = 1 square foot (*sq. ft.*)
9 or 3 × 3 square feet = 1 square yard (*sq. yd.*)
4840 square yards = 1 acre (*ac.*)
640 acres = 1 square mile (*sq. ml.*)

[40 poles (*pl.*) = 1 rood (*ro.*) 4 roods = 1 acre = 10 square chains.]

Solidity.

1728 or 12 × 12 × 12 cubic inches (*c. in.*) = 1 cubic foot (*c. ft.*)
27 or 3 × 3 × 3 cubic feet = 1 cubic yard (*c. yd.*)

Liquid Measure.

4 gills = 1 pint (*pt.*)
2 pints = 1 quart (*qt.*)
4 quarts = 1 *gallon* (*gal.*)

Dry Measure.

4 pecks (*pk.*) = 1 bushel (*bus.*)
8 bushels = 1 quarter (*qr.*)

[2 gallons = 1 peck.]

Time.

60 seconds (*sec.*) = 1 minute (*min.*)
60 minutes = 1 hour (*hr.*)
24 hours = 1 *day.*
365 days = 1 common year (*yr.*)
366 days = 1 leap year.

Digitized by Google

In the foregoing tables the standard units are printed in italics. Fractions are not used.

Other tables to be retained for reference merely are Apothecary's weight, Troy weight, angular measure, foreign measures, &c.

4. Constants of frequent occurrence should be remembered, such as the circumference of a circle is 3·1416 times its diameter nearly, the circumference of the Earth is 25000 miles nearly, 1 cubic foot of water weighs 1000 ounces nearly, &c.

Others of less frequent occurrence should be tabulated for reference, as, specific gravities of common substances, iron, lead, silver, gold, mercury, &c.; one gallon of water weighs 10 lb., and occupies 277·274 cubic inches nearly; fathom, knot; speed of light, of sound, of falling bodies, atmosphere as a unit of pressure, &c.

III. Processes.

5. It should be clearly understood that the fundamental operations of arithmetic are few. From addition and subtraction are developed multiplication and division, involution and evolution; and the numerous "Rules" in text-books are for the most part classified lists of examples of the application of these operations. As a matter of fact, the fundamental laws of arithmetic and algebra together are only three in number (see § 28).

6. *Decimals.*—It should be shown that the decimal notation for fractions (that is "Decimal Fractions") is merely an extension of the ordinary notation for integers.

7. *Metric System.*—The metric system should be taught, even if the time at the teacher's disposal is so small as to compel him to restrict his treatment to a single measure, for example, length. The principle of the system is too valuable to be passed over.

8. *Practice.*—The old rule of Practice should be omitted. In the cases where a simple fraction will suffice, retain such exercises as, to find the value of a number of articles whose price is a simple fraction of a pound, of a shilling, &c. These exercises can be introduced as applications of fractions.

Digitized by Google

9. *The Unitary Method and Proportion.*—Opinion is by no means unanimous as to the relative values of these two processes. There seems to be no doubt that the unitary method can be used with intelligence at an earlier age than that of proportion, but it should not supplant proportion throughout. The ideas involved in proportion are of more general importance than anything in the unitary method.

10. *Evolution.*—The extraction of square root should be taught as a part of arithmetic, because the need for it arises so frequently; the extraction of cube root should be treated as a part of algebra.

The following are useful approximations, especially the first two.

$$\sqrt{2} = \sqrt{(50/25)} = \sqrt{(49/25)} \text{ nearly} = 7/5 \text{ nearly.}$$
$$\sqrt{3} = \sqrt{(48/16)} = \sqrt{(49/16)} \text{ ,, } = 7/4 \text{ ,,}$$
$$\sqrt{5} = \sqrt{(80/16)} = \sqrt{(81/16)} \text{ ,, } = 9/4 \text{ ,,}$$
$$\sqrt{7} = \sqrt{(63/9)} = \sqrt{(64/9)} \text{ ,, } = 8/3 \text{ ,,}$$
$$\sqrt{11} = \sqrt{(99/9)} = \sqrt{(100/9)} \text{ ,, } = 10/3 \text{ ,,}$$

[*Note.* $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{9}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12} = 15/6$, 16/6, 17/6, 18/6, 19/6, 20/6, 21/6.]

IV. Computation.

11. Systematic endeavour should be made to develope in the pupil the power of accurate and rapid computation. This accomplishment can be attained by continued practice. It requires no severer mental strain than does the acquirement of dexterity in any mechanical exercise.

12. Beginners should be taught to compute in such a way that there may, if possible, be nothing to unlearn. "The act of addition must be made in the mind without assistance; you must not permit yourself to say, 4 and 7 are 11, 11 and 7 are 18, &c., but only 4, 11, 18, &c." "Learn the multiplication table so well as to name the product the instant the factors are seen; that is, until 8 and 7, or 7 and 8, suggest 56 at once, without the necessity of saying '7 times 8 are 56.'" (De Morgan's *Arithmetic*, 5th edition, pp. 162, 163.)

13. The addition table should be committed to memory as strictly as the multiplication table—both having been first of all constructed by the pupil, with the aid of counters or the like.

14. There should be systematic practice at exercises containing all the typical difficulties (compare the pianist's practice to acquire

Digitized by Google

technical dexterity.) For exercises of this kind see De Morgan (1) *Arithmetic*, 5th ed., appendix, (2) *Supplement to Penny Cyclopædia*, article *Computation*, (3) *British Almanac and Companion* for 1844.

15. As a further aid to rapid computation, the pupil should be exercised in certain labour-saving devices, namely, (1) methods to meet particular frequently-occurring cases (see § 16 following), and (2) the general abbreviated methods of working to produce results to any required degree of approximation (see §§ 18, 19).

16. *Short and easy methods.*—It is not suggested that all the methods in the following list (which does not pretend to be exhaustive) should be systematically taught.

Addition.—Example.—In adding two numbers, the first or second may be increased, and the second or first diminished to the same extent.

As, $57 + 39 = 60 + 36.$

$= 56 + 40.$

Or again, $57 + 39 = 57 + 30 + 9.$

$= 50 + 39 + 7.$

Subtraction.—Example.—In subtracting two numbers, both may be equally increased or diminished.

As, $57 - 39 = 58 - 40$

$346 - 328 = 46 - 28$

The following is frequently useful:

$100 - 58{\cdot}2476 = 41{\cdot}7524.$

Subtract, 5 from 9, 8 from 9, 2 from 9, &c., 6 from 10.

Multiplication.—When the multiplier ends with 5, in general double it, and halve the answer, or, when the multiplicand is even, halve the multiplicand before multiplying.

To multiply by 5, 5^2, 5^3, 5^4, &c.
multiply by 10, 10^2, 10^3, 10^4, &c.
and divide by 2, 2^2, 2^3, 2^4, &c.

To multiply by 9, 99, 999, &c.
multiply by 10, 100, 1000, &c.
and subtract the multiplicand. No working need be written down, only the answer.

To multiply by 11, 111, 1111, &c. If by 11, suppose a zero to be placed to right and left of the multiplicand. Beginning from right, add the digits in successive pairs. If by 111, suppose two zeros so placed, and add in successive threes, &c.

Digitized by Google

To multiply by 98, 97, 96, &c., multiply by 100, and subtract 2, 3, 4, &c., times the multiplicand. Similarly for 998, 997, &c.

To multiply by 42, 63, 189, &c., use the first line of the partial product instead of the multiplicand to get the second line.

To square a number of two digits. The identity

$a^2 = (a-b)(a+b)+b^2$, gives for example $58^2 = 56 \times 60 + 2^2$.

Division.—When the divisor ends in 5, in general double both divisor and dividend.

To divide by	$5,\ 5^2,\ 5^3,\ 5^4$, &c.
multiply by	$2,\ 2^2,\ 2^3,\ 2^4$, &c.
and divide by	$10,\ 10^2,\ 10^3,\ 10^4$, &c.

To find the value of 12 articles, given the price of one. Consider the pence in the price as shillings.

To find the value of 20 articles, given the price of one. Consider the shillings in the price as pounds.

To find the value of 100 articles, given the price of one. For every farthing take as many pence and twice as many shillings.

To find the price of 112 pounds, given the price of one pound. Add the price of 12 to the price of 100.

Percentage.—To find 5 % of a sum of money, take 1s. for every £1 in the sum. For $2\frac{1}{2}$ % take 6d. for every £1.

Five per cent. per annum is a penny per pound per month. Similarly for two and a half per cent.

17. Practice in the approximate methods should be kept up from the earliest acquaintance with the extended decimal notation. The young pupil, instead of being pushed through the weary and unprofitable mazes of circulating decimals, should rather be exercised in the metric system, in approximate addition, subtraction, multiplication, and division, and in learning to turn the decimal system to account in treating questions of British money.

18. The abbreviated methods of multiplication and division are now given in all good text books, and therefore need not be treated in detail here. The following recommendations are made.

a. In contracted multiplication the multiplier need not be written reversed. This procedure, though a useful and obvious expedient to the mature judgment, sometimes puzzles beginners. The pupil should be accustomed to multiply, beginning at the left as well

Digitized by Google

as at the right, and if the units, tens ... digits of the multiplier be placed under the units, tens ... digits of the multiplicand, the pupil easily learns to take the figures of the multiplier in *any order whatever*, and to place the product correctly. When this point has been reached, he may be allowed to carry on multiplication from the right in the usual way, and then by applying the principle involved in the statement that "a 3rd place decimal, multiplied by a 2nd place decimal, gives a 5th place decimal," he readily learns where he should begin to multiply in order to obtain the requisite approximation.

(*b*) In performing any long division the pupil should be required to get the successive remainders without writing down the partial products. This is quite easy, even without the special preparation recommended in § 14.

Thus the procedure to get the last remainder in the annexed example, stated in its fullest form for the sake of illustration, is:—

4 times 7, 28 and 9 (write 9 down) 37, carry 3; 4 times 5, 20, and 3 carried, 23 and 1 (write 1 down) 24, carry 2; 4 times 3, 12 and 2 carried, 14, and 2 (write 2 down) 16.

```
357)83757(234
    1235
     1647
      219
```

19. To express shillings, pence, and farthings as the decimal of a pound, note that

$2/ = \cdot 1$ of £1.
$1/ = \cdot 05$ of £1.
$\frac{1}{4}$d. $= \cdot 001 + 1/24$ of $\cdot 001$ of £1.

Hence the following rule, which gives the result correct to the thousandth of a £: Call the florins tenths, the odd shilling (if any) five hundredths; for the thousandths reduce the rest to farthings, adding 1 if there be more than 12, 2 if more than 36.

Examples: (1) £2 ,, 7 ,, $1\frac{3}{4}$

3fl. = ·3
1s. = ·05
7f. = ·007

£2·357

Digitized by Google

(2) £3 „ 16 „ 5½

8fl. = ·8

21f. = ·022

£3·822.

(3) £4 „ 15 „ 10½

7fl. = ·7

1s. = ·05

41f. = ·043

£4·793.

The inverse operation is as follows:—

(1) £7·293 = £7·2 + ·05 + ·043.

= £7 „ 4/ + 1/ + 41f.

= £7 „ 5 „ 10½.

(2) £4.567 = £4·5 + ·05 + ·017.

= £4 „ 11 „ 4.

Verifiable exercises on the above can be got at once from the ordinary examples of the compound rules. Turn a column of a compound addition sum into decimals, add the lines as decimals, convert the answer into £ s. d., and compare the result with that obtained in the usual way. Similarly with subtraction, multiplication, and division. The pupil will see how the absolute error increases in multiplication and diminishes in division, the relative error remaining the same.

20. *Logarithms.*—The last matter requiring discussion under the head of computation is the question of teaching logarithms.

It is recommended that a knowledge of logarithms be considered a necessary part of a complete arithmetical training. The use of logarithms with a sufficient amount of theory can be taught without the aid of advanced algebra.

V. Theory.

21. The necessity of developing the theory of arithmetic, in immediate connection with the practice of the special processes is universally acknowledged, but there is some difficulty in effecting this. A few suggestions are offered on points where a knowledge of theory is most frequently found wanting.

Digitized by Google

22. To recommend any text-books in use in this country would be invidious and perhaps useless, but such objection does not apply to foreign treatises. The two following may be mentioned as affording instances of the adaptation to arithmetic of theory which is usually left to algebra:—

(1.) *Traité Élémentaire d'Arithmétique*, par J. F. J. Kleyer [part 1 (75 centimes), part 2 (75 centimes), part 3 (1·10 francs)]. H. Dessain, Liége, rue Trappe, 7. 1885–6.

(2.) *Traité d'Arithmétique Élémentaire*, par L'Abbé E. Gelin (5 francs), Paris, Librairie du *Journal de Mathématiques Élémentaires*, rue des Écoles, 17. 1886.

23. *Scales of Notation.*—The theory of other scales of notation may be introduced to illustrate the decimal scale, if only to show the pupil that there is nothing essential in the selection of 10 as the radix of the ordinary scale. The Roman notation should be explained, and examples given in writing and reading it.

24. *Divisibility.*—The tests of divisibility of numbers by 2, 3, &c., should be given and proved.

25. *Greatest Common Measure.*—The theory of the method of finding the G. C. M. by reduction to primes, and of the ordinary method, should be explained.

This last may be done by means of the two following theorems, of which proofs may be given that are perfectly general in method, although particular numbers may be employed.

a. Every common measure of two numbers, say 36 and 28, is a measure of their difference 8 (and also of their sum 64).

Thus every common measure of 36 and 28 is a common measure of 28 and 8.

b. Conversely, every common measure of the less of two numbers 28 and their difference 8 is a measure of the greater of the numbers 36.

Thus any common measure of 28 and 8 is a common measure of 36 and 28.

From these two theorems it follows that the problem of finding the G. C. M. of the numbers 36 and 28 may be reduced to the simpler problem of finding the G. C. M. of the numbers 28 and 8, and this again to that of finding the G. C. M. of 20 and 8, and so on till the G. C. M. is found.

Digitized by Google

Moreover, instead of taking the difference of, say, 28 and 8, then the difference of 20 and 8, and again the difference of 12 and 8, we may take the difference of 28 and any multiple of 8, for instance, the difference of 28 and 8×3 or the difference of 28 and 8×4.

26. *Proportion.*—The following is given as a very brief outline, sufficient for arithmetic, of the theory of proportion. In the proofs letters are used, but they are meant to be given alongside of full numerical exemplification (see § 29).

Definition 1.—The ratio of one whole number to another is measured by the fraction which the one is of the other; and the ratio of one quantity to another is the ratio of the two whole numbers that express these quantities in terms of the same unit.

Definition 2.—When two ratios are equal, the four quantities form a proportion.

Theorem 1.—If the fourth term of a proportion be greater than the third, the second shall be greater than the first, if equal equal, and if less less.

Let $a : b = c : d$ be the proportion.

Because $a : b = c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$.

Now (1) if d be greater than c,

then the fraction $\frac{c}{d}$ is less than 1;

therefore the fraction $\frac{a}{b}$ is less than 1;

therefore b is greater than a.

(2) If d be equal to c,

then the fraction $\frac{c}{d}$ is equal to 1;

therefore the fraction $\frac{a}{b}$ is equal to 1;

therefore b is equal to a.

(3) If d be less than c,

then the fraction $\frac{c}{d}$ is greater than 1;

therefore the fraction $\frac{a}{b}$ is greater than 1;

therefore b is less than a.

Digitized by Google

Theorem 2.—In a proportion the product of the first and fourth terms is equal to the product of the second and third.

Let $a : b = c : d$ be the proportion.

Because $a : b = c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$;

therefore $\frac{a}{b} \times b \times d = \frac{c}{d} \times d \times b$;

therefore $a \times d = b \times c$.

Application—First.—Theorem 1 enables us to arrange in a proper order any four quantities among which a proportion is assumed to exist.

Example.—If 2 lb. of tea cost 5/, what will 6 lb. cost?

It is assumed that the ratio of 2 lb. to 6 lb. is equal to the ratio of the price of 2 lb. to the price of 6 lb.

Let x/ denote the unknown price, which it is usual to write as the fourth term of the proportion. Then the third term will be 5/, for a ratio can exist only between quantities of the same kind.

To ascertain whether 2 lb. or 6 lb. should occupy the second place, we inquire whether x/ is greater than 5/, equal to it, or less than it. On consideration of the question it will be seen that x/ must be greater than 5/, since it is the price of a larger quantity; that is, the fourth term of the proportion is greater than the third. Hence the second term must be greater than the first; therefore 6 lb. must be the second term, and 2 lb. the first.

Second.—Theorem 2 enables us to supply any term of a proportion which may be omitted.

Example.—In the proportion 2 lb : 6 lb. $= 5/ : x/$, supply the omitted term.

The proportion $2 \text{ lb.} : 6 \text{ lb.} = 5/ : x/$

may be written $2 : 6 = 5 : x$;

therefore $2 \times x = 6 \times 5$;

therefore $x = \frac{6 \times 5}{2} = 15.$

Hence the omitted term is 15/.

Digitized by Google

VI. Links.

27. The recommendations under this heading refer more to the spirit than to the substance of the teaching of arithmetic, and definite suggestions must be few. The teacher of arithmetic who knows, at least, algebra and geometry, and is not tied down to the notion that things with different names must be kept severely apart, no matter how close their relation, can always be trusted to teach arithmetic so as to make prominent those general ideas which, though fully developed only in the higher branches of mathematics, are already of fundamental importance in elementary arithmetic.

28. *The Three Fundamental Laws.*—In any attempt to deal with arithmetic as a science, an endeavour should be made to show the pervading importance of the laws of commutation, association, and distribution. With a little skill in the introduction of these notions and a word in season to keep them alive, it is by no means difficult to obtain satisfactory results even with young pupils. The pupils should gain familiarity with these laws as exemplified in addition, subtraction, multiplication, and division. At every stage of an example (say of reducing a particular fractional expression) they should occasionally be required to refer the process to the fundamental law under which it comes.

The development of the science of algebra from these laws has been carried out in the best recent text books, which may be consulted for full details; while the same has been done for arithmetic by Homersham Cox (*Principles of Arithmetic*, Deighton, Bell & Co., Cambridge, 1885).

29. *Generalised Examples.*—It is recommended that examples in arithmetic should be systematically generalised. Thus, following a series of such examples as: If 2 lb. of tea cost 5/ what will 6 lb. cost? there should be given such as: If p lb. of tea cost q/ what will r lb. cost? and then it should be shown how the answer to such a general question contains as particular cases the results of all numerical examples of the same class. Work of this kind ensures the intelligent use of the formulae commonly used in problems of interest and the like.

Digitized by Google

30. *Mensuration.*—Thoroughly practical examples in mensuration, derived from actual measurement of the school surroundings, with the making of actual plans, can be made a sound basis for geometry, when the pupils come to study it. Whether arithmetical illustrations be part of the teaching of geometry, or geometrical notions be drawn on for examples in arithmetic, is immaterial; but the two subjects should certainly be made to lend mutual aid, as here suggested.

VII. General Hints.

31. Do not hurry from stage to stage in arithmetic at the sacrifice of thoroughness. It is not necessary that a pupil of thirteen should be familiar with stocks and shares.

32. Give many easy examples illustrating principles at the introduction of every new process. This is the best kind of "mental arithmetic."

33. In problems involving a number of multiplications and divisions, insist on having all the steps indicated before any "working out" is attempted. In most cases this leads to great simplification.

34. Encourage the pupil always to make a rough forecast of the result. (To originate this habit it will be found a help to make the pupil write his rough estimate on one side of his slate before proceeding to find the exact result on the other.) The habit of doing this is beneficial in two ways:—

a. It shows the importance of approximate methods generally, and in particular of beginning in multiplication at the left of the multiplier.

b. It will prevent the pupil from resting content with a ridiculous answer.

35. Some method of encouraging speed should be adopted, such as numbering or rearranging the pupils as they finish working. The names of the first half dozen written on the blackboard, even if no additional marks be allowed them, stimulates activity. This plan should, of course, be adopted only at intervals.

Digitized by Google

36. An attempt should be made to give the pupils some idea of the magnitude of the units of length, weight, &c. Thus, diagrams of the units of the metric system should be placed on the walls of the schoolroom; measurements of the classroom, blackboards, &c., might be made and recorded where they can be readily seen. The weights and dimensions of the slates and the text-books might be ascertained.

37. In practical applications of arithmetic (to problems of money, &c.) some care should be taken that the problems are such as actually do occur. For example, it should be borne in mind that income-tax is never calculated on odd shillings, that the Post Office Savings Bank does not allow interest for a fraction of a month, that banks do not deal in fractions of a penny, and so on.

In teaching "Stocks" the money column of the newspaper should be explained, and be made to furnish problems. It may be mentioned that share lists giving particulars (amount paid up per share, &c.) not contained in the newspapers, can be procured from stock-brokers, or from banks.

For other practical problems geometry and physics should be drawn upon as largely as possible.

38. An operation or a notion should be taught along with its inverse, as, addition with subtraction, multiplication with division, power with root, measure with multiple, &c.

39. As far as exigencies of examinations will allow, postpone formal algebra. The present tendency is to begin algebra too soon by a year or two.

Digitized by Google

Edinburgh Mathematical Society.

LIST OF MEMBERS.

SIXTH SESSION, 1887–88.

Peter Adam, M.A., 81 Claremont Street, Glasgow.
Peter Alexander, M.A., 16 Smith Street, Hillhead, Glasgow.
John Alison, M.A., George Watson's College, Edinburgh (*Hon. Treasurer*).
R. E. Allardice, M.A., F.R.S.E., Assistant to the Professor of Mathematics, University, Edinburgh.
P. J. Anderson, M.A., LL.B., Lecturer, Church of Scotland Training College, Aberdeen.
A. H. Anglin, M.A., LL.D., F.R.S.E., M.R.I.A., Professor of Mathematics, Queen's College, Cork.
Paul Aubert, 14 Rue de l'Arsenal, Rochefort-sur-Mer, Charente Inférieure, France.
A. J. G. Barclay, M.A., F.R.S.E., George Watson's College, Edinburgh.
C. W. C. Barlow, M.A., Rochester House, Little Ealing, London, W.
Peter Bennett, Glasgow and West of Scotland Technical College, Andersonian Buildings, Glasgow.
James Blyth, M.A., F.R.S.E., Professor of Physics and Mathematics, Andersonian College, Glasgow.
James Bolam, Government Navigation School, Leith.
William Brown, Merchant Taylors' School, Charterhouse, London.
Rev. H. H. Browning, B.D., 128 Byres' Road, Glasgow.

Digitized by Google

James Buchanan, M.A., Peterhouse, Cambridge.
J. R. Burgess, M.A., Mathematical Master, Merchiston Castle, Edinburgh.
John Watt Butters, M.A., 13 M'Dowall Street, Edinburgh.
John Cameron, Mathematical Master, Academy, Montrose.
Peter Campbell, M.A., Musselburgh.
Charles Chree, M.A., King's College, Cambridge.
George Chrystal, M.A., LL.D., F.R.S.E., Professor of Mathematics, University, Edinburgh (*Hon. Member*).
John Clark, M.A., Pembroke College, Cambridge.
John Brown Clarke, Heriot's Hospital School, Edinburgh,
Walter Colquhoun, Garnethill School, Glasgow.
G. P. Lenox Conyngham, Lieutenant, R.E., India.
James J. Craig, 1 Leven Terrace, Edinburgh.
John Craig, M.A., Heriot's Hospital School, Edinburgh.
Lawrence Crawford, M.A., King's College, Cambridge.
H. G. Creelman, B.A., B.Sc., 7 Parkside Terrace, Edinburgh.
J. Crockett, M.A., 10 Gillespie Crescent, Edinburgh.
R. J. Dallas, M.A., King's College, Cambridge.
Hugh Dickie, B.A., Rector, Academy, Kilmarnock.
H. N. Dickson, F.R.S.E., 38 York Place, Edinburgh.
J. D. H. Dickson, M.A., F.R.S.E., Peterhouse, Cambridge.
Rev. W. Cooper Dickson, M.A., U.P. Manse, Muckart, by Dollar.
J. Macalister Dodds, M.A., Peterhouse, Cambridge.
Alex. Don, M.A., Waid Academy, Anstruther.
Alex. B. Don, M.A., High School, Dunfermline.
John Don, M.A., Rainings' School, Inverness.
John Dougall, M.A., Kippen, Stirling.
Rev. James Edwards, M.A., B.Sc., Professor of Mathematics, Stellenbosch, South Africa.
Arch. C. Elliott, B.Sc., Assistant to the Professor of Engineering, University, Edinburgh.
Jas. Gibson Fairweather, B.Sc., C.E., 8 Findhorn Place, Edinburgh.
R. M. Ferguson, Ph.D., F.R S.E., 8 Queen Street, Edinburgh.
Rev. Alex. M. Forbes, M.A, Free Church Manse, Towie, Aberdeenshire.

Digitized by Google

A. Y. Fraser, M.A., F.R.S.E., Heriot's Hospital School, Edinburgh (*Hon. Secretary*).

John S. Fraser, M.A., Solicitor, 28 High Street, Inverness.

Norman Fraser, M.A., 25 Warrender Park Road, Edinburgh.

E. P. Frederick, M.A., Loretto, Musselburgh.

George A. Gibson, M.A., Assistant to the Professor of Mathematics, University, Glasgow (*Vice-President*).

Charles Gordon, F.F.A., F.I.A., Secretary, Scottish Provincial Insurance Company, Union Street, Aberdeen.

R. P. Hardie, 4 Scotland Street, Edinburgh.

William Harvey, B.A., LL.B., 53 Castle Street, Edinburgh.

J. Holliday, 4 Gillespie Crescent, Edinburgh.

James Irvine, M.A., B.Sc., 31 Warrender Park Terrace, Edinburgh.

John G. Kerr, M.A., Church of Scotland Training College, Glasgow

Andrew King, M.A., 4 Panmure Place, Edinburgh.

John King, M.A., George Watson's College, Edinburgh.

Cargill C. Knott, D.Sc., F.R.S.E., Professor of Physics, Imperial College, Tokio, Japan.

R. Scott Lang, M.A., Professor of Mathematics, University, St Andrews.

A. P. Laurie, B.A., Nairne Lodge, Duddingston.

John Lockie, Consulting Engineer, 2 Custom House Chambers, Leith.

J. Barrie Low, M.A., 17 Elgin Terrace, Dowanhill, Glasgow.

John Mack, M.A., Douglas Cottage, Baillieston, Glasgow.

John M'Cowan, M.A., Royal College of Science, Dublin.

Farquhar Macdonald, M.A., High School, Glasgow.

Hector M. Macdonald, M.A., Clare College, Cambridge.

J. C. R. Macdonald, W.S., Dumfries.

W. J. Macdonald, M.A., F.R.S.E., Daniel Stewart's College, The Dean, Edinburgh (*President*).

A. Macfarlane, D.Sc., F.R.S.E., University, Austin, Texas, U.S.A.

J. S. Mackay, M.A., LL.D., F.R.S.E., 69 Northumberland Street, Edinburgh.

William Mackay, M.A., Rector, Academy, Rothesay.

J. L. Mackenzie, M.A., Raeden House, Aberdeen.

Digitized by Google

JOHN MACKENZIE, 3 Valleyfield Street, Edinburgh.

P. MACKINLAY, M.A., Rector, Church of Scotland Training College, Edinburgh.

MAGNUS MACLEAN, Natural Philosophy Laboratory, University, Glasgow.

BASIL S. M'LELLAN, M.A., Gordon's College, Aberdeen.

CHARLES M'LEOD, M.A., Assistant to the Professor of Mathematics, University, Aberdeen.

DONALD MACMILLAN, M.A., Clifton Bank School, St Andrews.

JAMES MACMILLAN, M.A., 362 St Vincent Street, Glasgow.

J. MACMILLAN, M.A., B.Sc., F.R.S.E., Academy, Perth.

A. J. MACNICOL, M.A., Student, Dunoon.

Rev. J. GORDON MACPHERSON, M.A., Ph.D., F.R.S.E., Ruthven Manse, Meigle, Perthshire.

JOHN M'WILLIAM, M.A., Schoolhouse, Insch.

ARTEMAS MARTIN, LL.D., U.S. Coast Survey Office, Washington, D.C., U.S.A.

Rev. CHARLES M. MIDDLETON, M.A., Edinburgh.

W. J. MILLAR, C.E., 261 West George Street, Glasgow.

ANDREW MILLER, M.A., High School, Dundee.

T. HUGH MILLER, M.A., 90 Belmont Gardens, Glasgow.

WILLIAM MILNE, M.A., F.R.S.E., High School, Glasgow.

W. T. MITCHELL, M.A., 20 Ampton Street, London, W.C.

ALEXANDER MORGAN, M.A., Church of Scotland Training College, Edinburgh.

J. T. MORRISON, M.A., Heriot-Watt College, Chambers Street, Edinburgh.

ANDREW MUIR, Academy, Glasgow.

THOMAS MUIR, M.A., LL.D., F.R.S.E., High School, Glasgow.

R. FRANKLIN MUIRHEAD, M.A., B.Sc., Rochester House, Ealing, London, W.

DAVID MUNN, F.R.S.E., 2 Ramsay Gardens, Edinburgh.

DAVID MURRAY, M.A., B.Sc., High School, Kilmarnock.

CHARLES NIVEN, M.A., D.Sc., F.R.S., Professor of Natural Philosophy, University, Aberdeen (*Hon. Member*).

F. GRANT OGILVIE, M.A., B.Sc., F.R.S.E., Principal, Heriot-Watt College, Chambers Street, Edinburgh.

R. T. OMOND, The Observatory, Ben Nevis.

Digitized by Google

WILLIAM PEDDIE, D.Sc., F.R.S.E., Natural Philosophy Laboratory, University, Edinburgh.
FLORA PHILIP, 8 Dean Terrace, Edinburgh.
ROBERT PHILIP, M.A., Hutcheson's Grammar School, Glasgow.
R. H. PINKERTON, Hogganfield, Glasgow.
CHARLES PRENTICE, C.A., 40 Castle Street, Edinburgh.
HARRY RAINY, M.A., 25 George Square, Edinburgh.
WILLIAM RAITT, M.A., B.Sc., 6 Kyle Park, Uddingston.
THOS. T. RANKIN, C.E., Gartsherrie Science Schools, Coatbridge.
WILLIAM REID, M.A., High School, Glasgow.
DAVID RENNET, LL.D., 12 Golden Square, Aberdeen.
JAMES RENTON, C.E., 19 Blackwood Crescent, Edinburgh.
WILLIAM RENTON, Ambleside.
D. RINTOUL, M.A., Clifton College, Bristol.
Rev. R. S. RITCHIE, Manse of Sorn, Ayrshire.
WILLIAM RITCHIE, George Watson's College, Edinburgh.
ALEX. ROBERTSON, M.A., 30A St Andrew Square, Edinburgh.
ROBT. ROBERTSON, M.A., Ladies' College, Queen St., Edinburgh.
H. C. ROBSON, M.A., 134 Kensington Park Road, London, W.
ALEXANDER RUSSELL, M.A., Caius College, Cambridge.
R. F. SCOTT, M.A., St John's College, Cambridge.
WILLIAM SCOTT, M.A., Schoolhouse, New Machar, Aberdeen.
E. J. SMITH, M.A., Royal High School, Edinburgh.
T. B. SPRAGUE, M.A., F.R.S.E., 29 Buckingham Terrace, Edinburgh.
G. N. STEWART, M.A., Owen's College, Manchester.
JAMES STRACHAN, M.A., Garnethill School, Glasgow.
P. G. TAIT, M.A., Sec.R.S.E., Professor of Natural Philosophy, University, Edinburgh (*Hon Member*).
JAMES TAYLOR, Academy, Edinburgh.
JAMES TAYLOR, M.A., The Institution, Dollar.
GEORGE THOM, M.A., LL.D., The Institution, Dollar.
ANDREW THOMSON, M.A., D.Sc., F.R.S.E., 1 Blackness Crescent, Dundee.
JAMES THOMSON, M.A., The Academy, Ayr.
Sir WILLIAM THOMSON, LL.D., D.C.L., F.R.S.S., L. and E., &c., Professor of Natural Philosophy, University, Glasgow (*Hon. Member*).

Digitized by Google

RICHARD THORNTON, M.A., Harris Academy, Dundee.

135 DAVID TRAILL, M.A., B.Sc., 14 Livingston Place, Edinburgh.

ROBERT TUCKER, M.A., Hon. Sec. London Mathematical Society, 27 Cantlowes Road, Camden Square, London, N.W.

JOHN E. VERNON, M.A., Arbirlot Free Church Manse, Arbroath.

JAMES WALKER, 3 Warrender Park Terrace, Edinburgh (*last known address*).

SAMUEL WALKER, M.A., B.Sc., Heriot's Hospital School, Edinburgh.

140 JOHN C. WATT, M.A., Jesus College, Cambridge.

JOHN WEIR, M.A., Mathematical Department, University, Glasgow.

WILLIAM WELSH, M.A., Jesus College, Cambridge.

Rev. THOMAS WHITE, B.D., 7 Regent Terrace, Edinburgh.

LAURENCE WIGHT, M.A., Lauder Road, Edinburgh.

145 A. M. WILLIAMS, M.A., 38 Fountainhall Road, Aberdeen.

Rev. JOHN WILSON, M.A., F.R.S.E., 27 Buccleuch Place, Edinburgh.

PHILIP WOOD, M.A., Grammar School, Darlington.

148 WILLIAM WYPER, C.E., 7 Bowmont Gardens, Hillhead, Glasgow.

Proposed at the June Meeting, for Election at the November Meeting, the following :—

D. F. LOWE, M.A., F.R.S.E., Headmaster, Heriot's Hospital School, Edinburgh.

ASUTOSH MUKHOPADHYAY, M.A., F.R.A.S., F.R.S.E., &c., Bhowanipore, Calcutta.

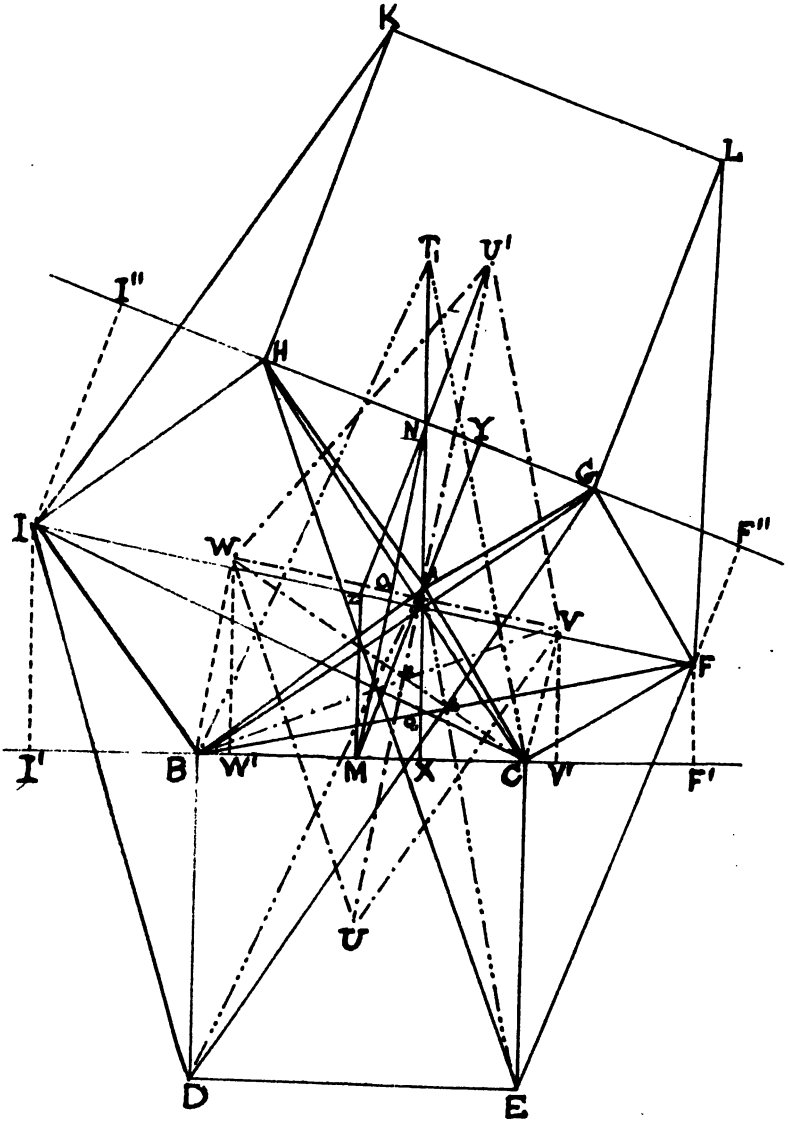
Rev. W. A. REID, M.A., 75 Union Street, Aberdeen.

WILLIAM WILSON, M.A., Heriot's Hospital School, Edinburgh.

Digitized by Google

Digitized by Google

Fig 1



Digitized by Google

Fig 2

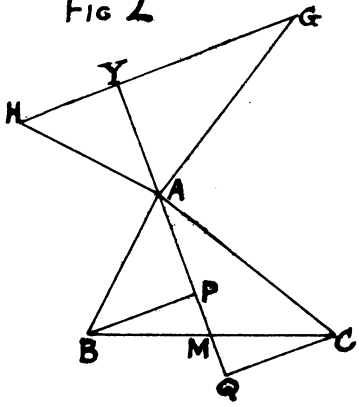


Fig 3

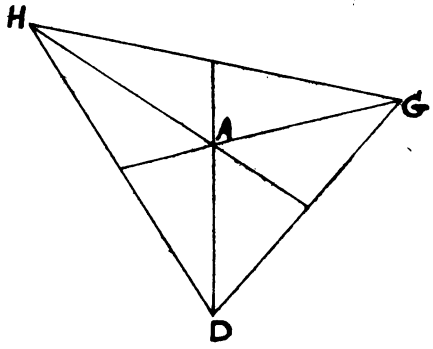


Fig 4

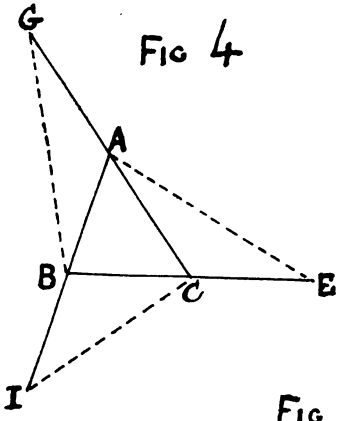


Fig 5

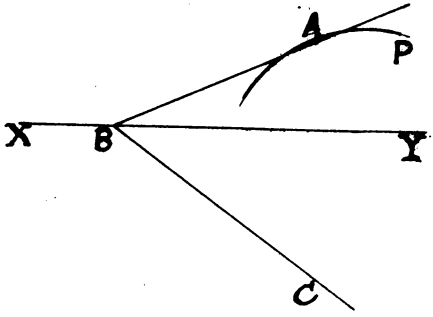


Fig 6

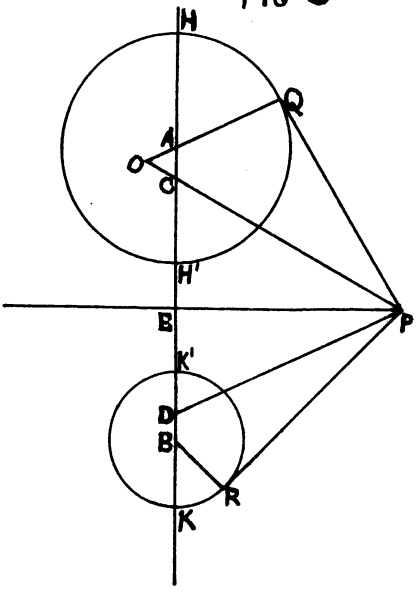
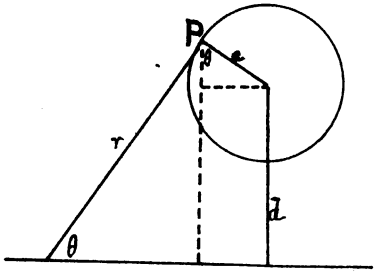


Fig 7



Digitized by Google

Digitized by Google

Digitized by Google

Fig 8

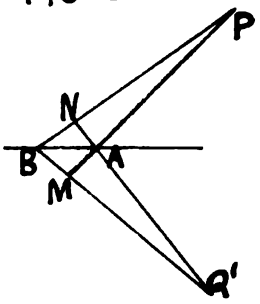


Fig 9

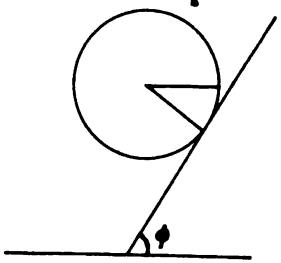


Fig 10

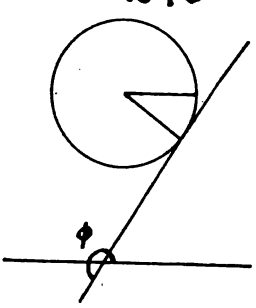


Fig 11

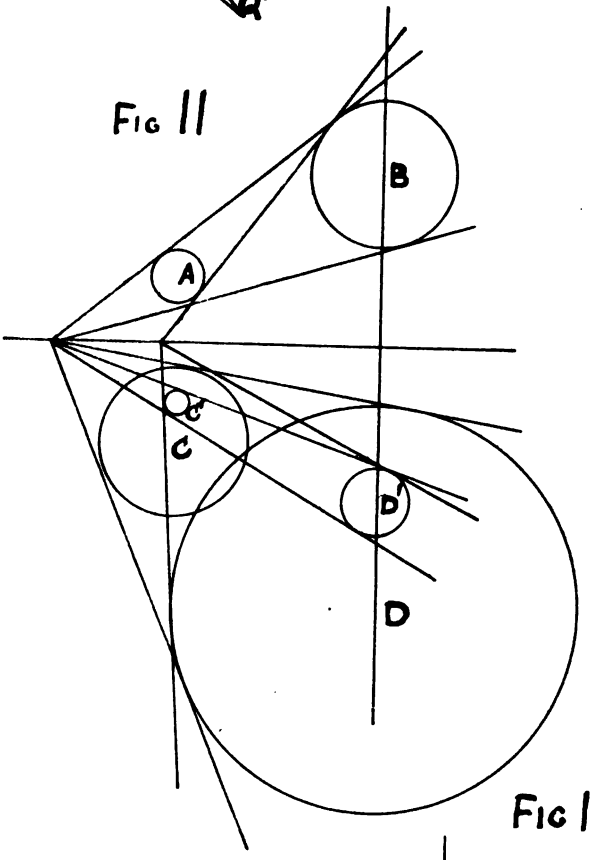


Fig 13

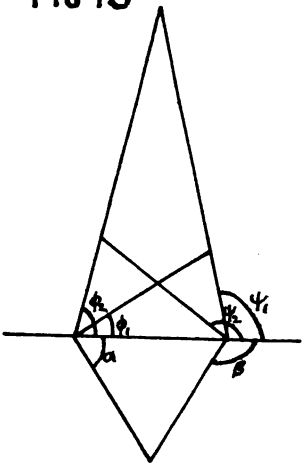
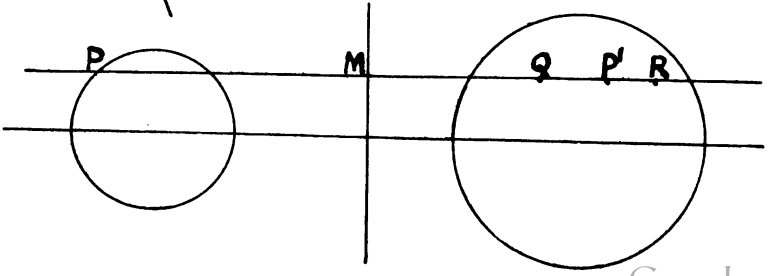


Fig 12



Digitized by Google

Fig 14

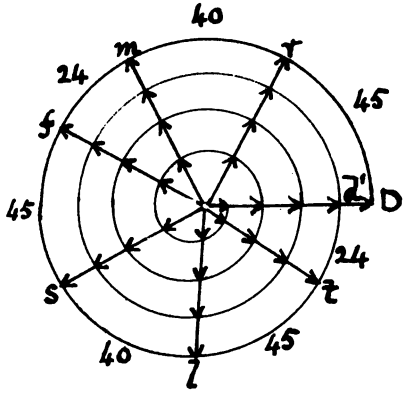


Fig 15

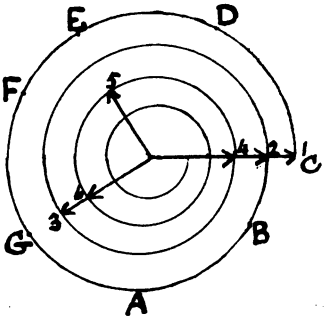


Fig 16

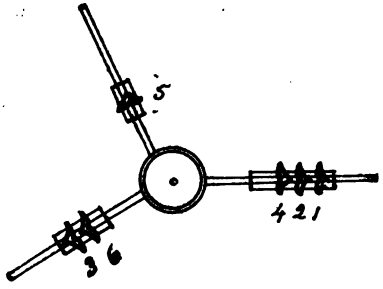
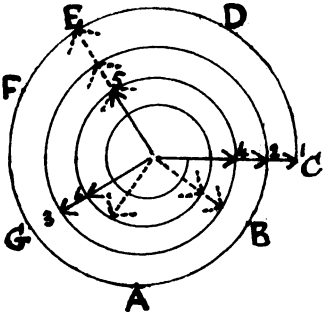


Fig 17



Digitized by Google

Digitized by Google

Digitized by Google

Fig 18

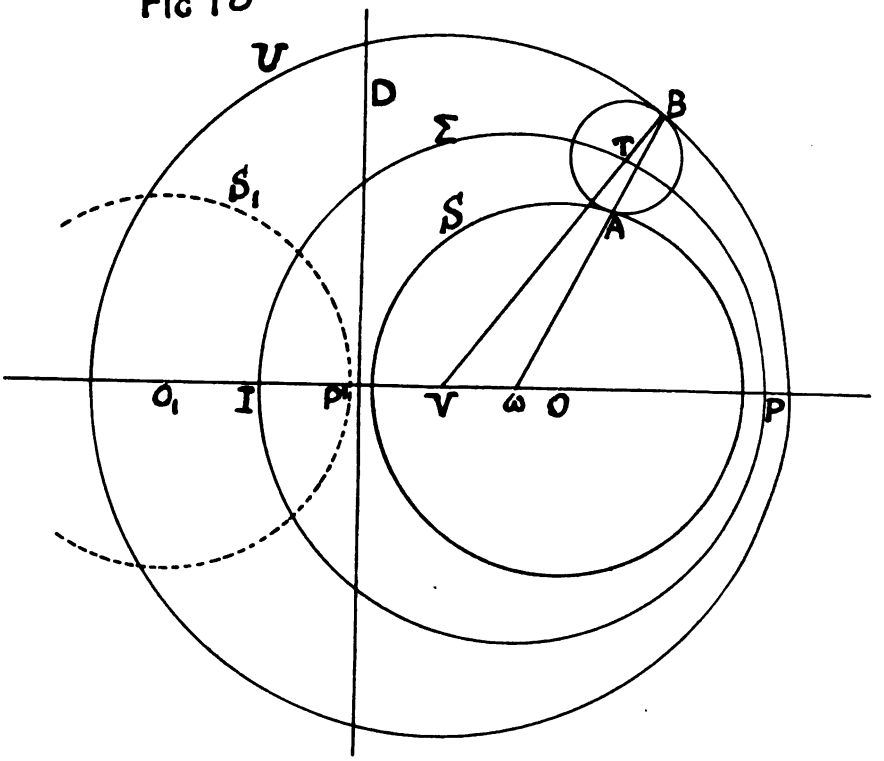
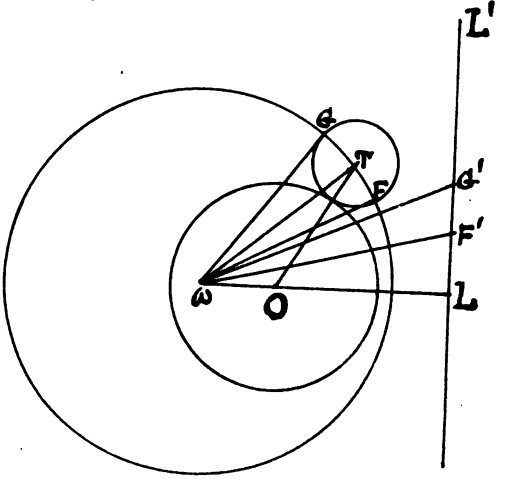
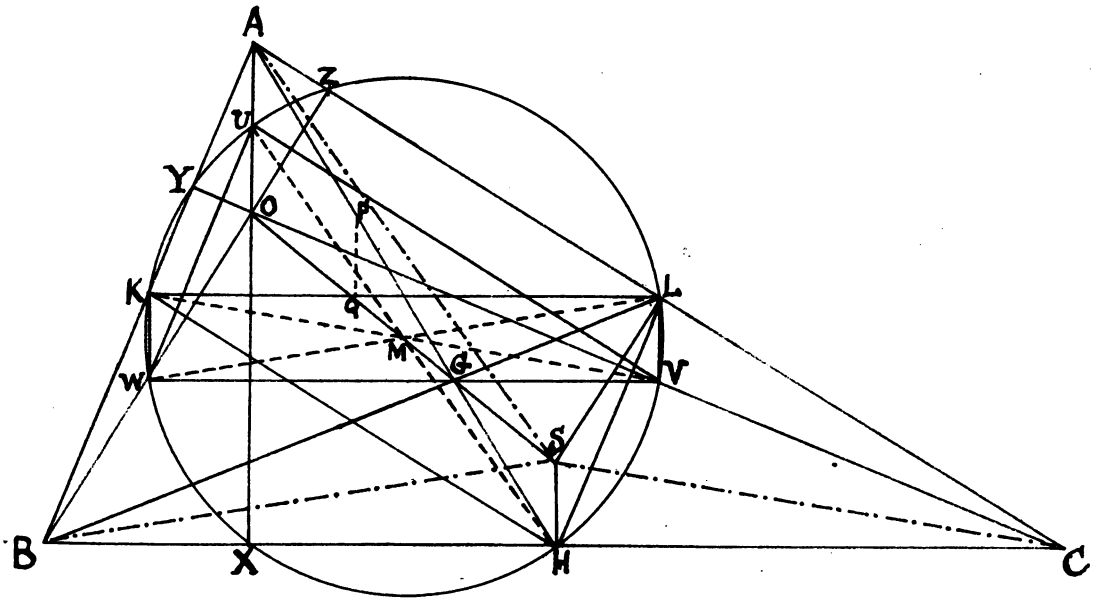


Fig 19



Digitized by Google

Fig 20



Digitized by Google

Digitized by Google

Digitized by Google

Fig 21

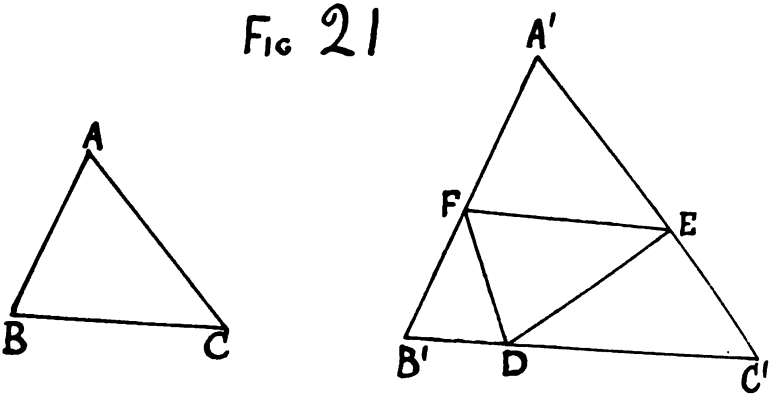


Fig 22

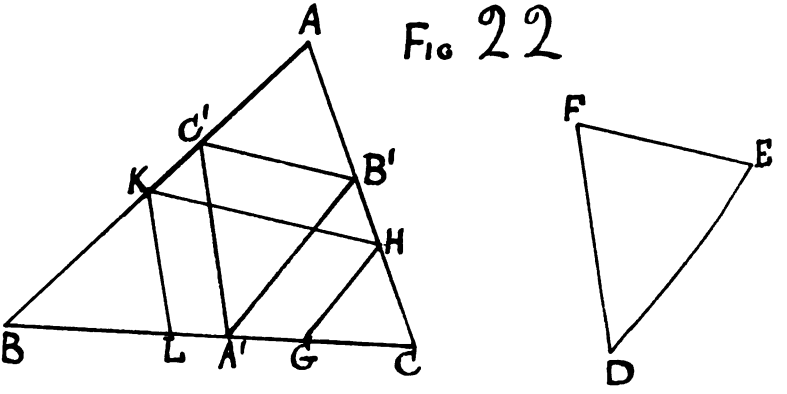


Fig 23

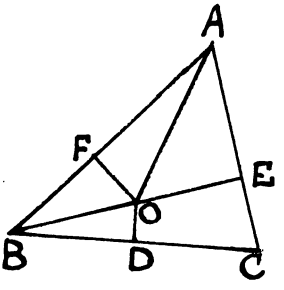
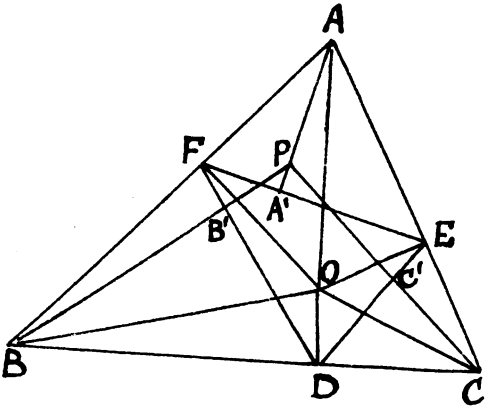


Fig 24



Digitized by Google

Fig 25

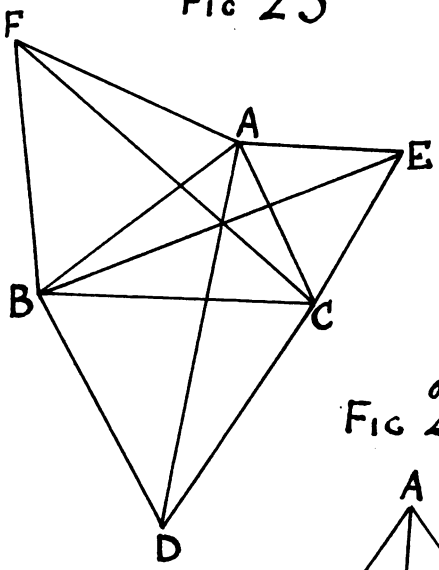


Fig 26

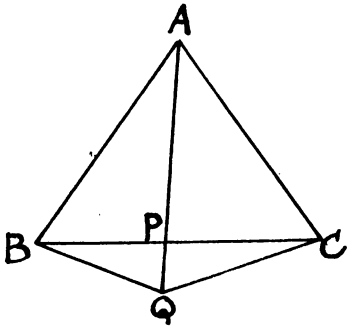
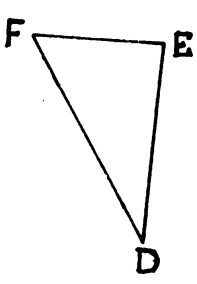
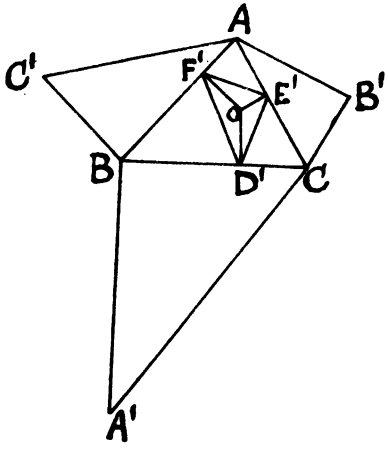


Fig 27





Digitized by Google

Digitized by Google

Fig 28

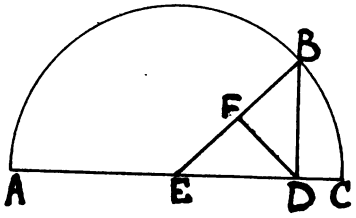


Fig 29

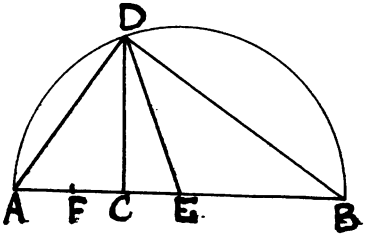


Fig 30

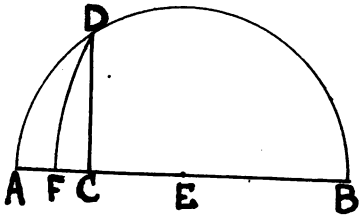


Fig 31

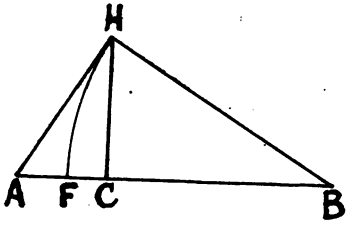


Fig 32

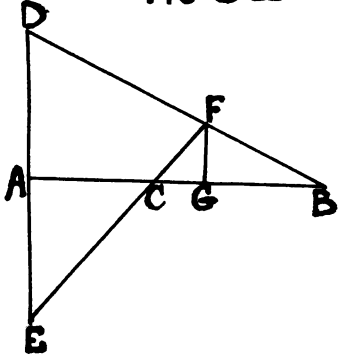


Fig 33

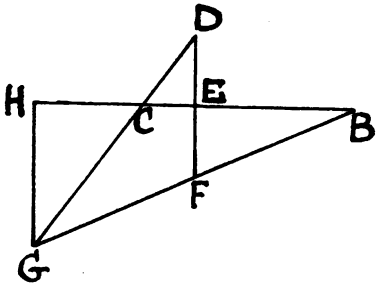


Fig 34

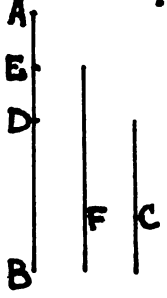


Fig 35

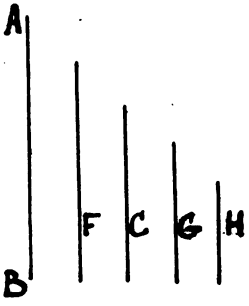
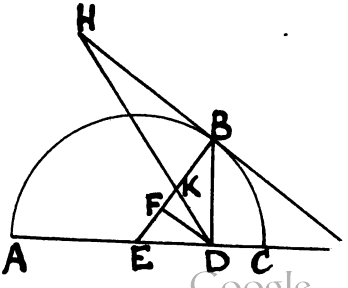


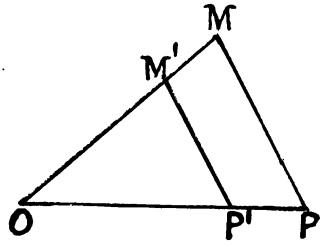
Fig 36



Digitized by Google

Digitized by Google

Fig 37



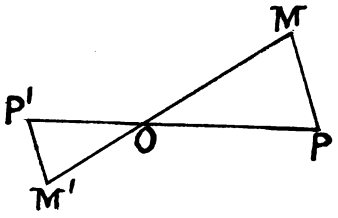
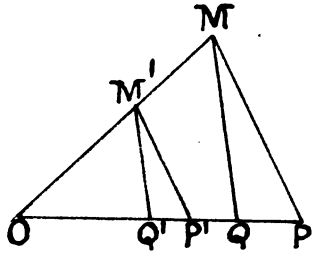


Fig 38



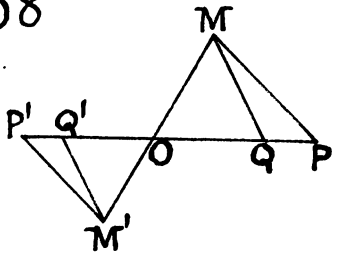
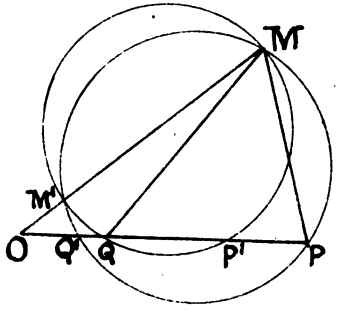


Fig 39



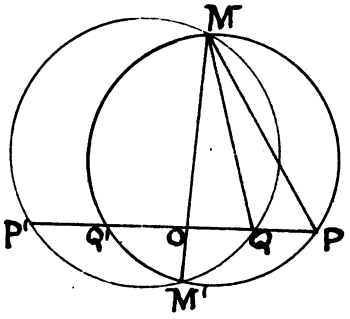


Fig 40

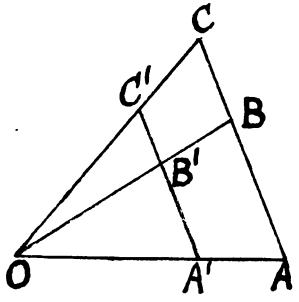
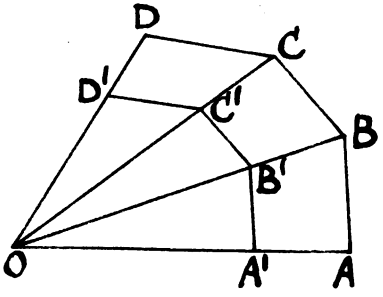


Fig 41



Digitized by Google

Digitized by Google

Fig 42

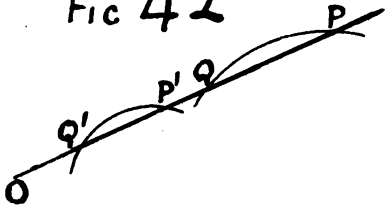


Fig 43

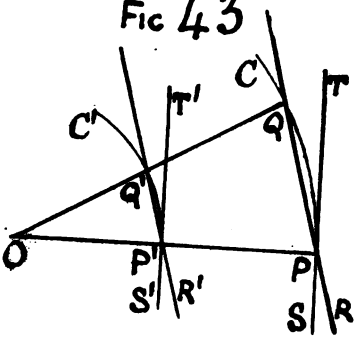


Fig 44

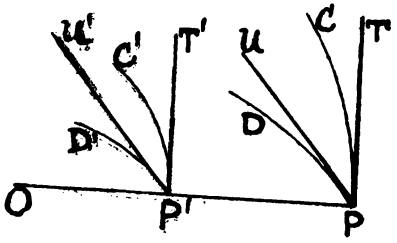
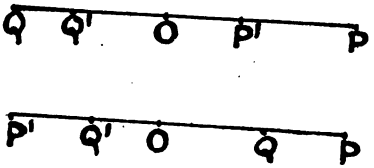


Fig 45



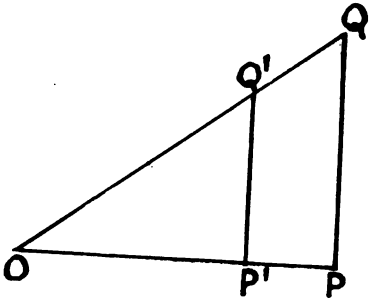


Fig 46

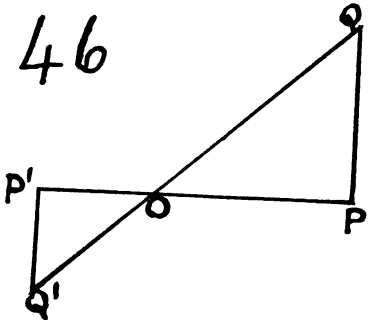
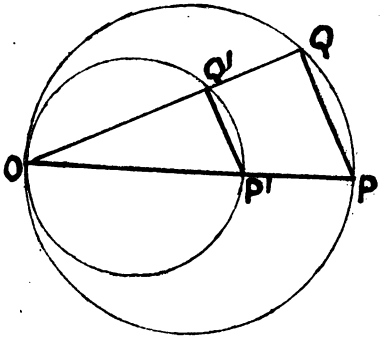
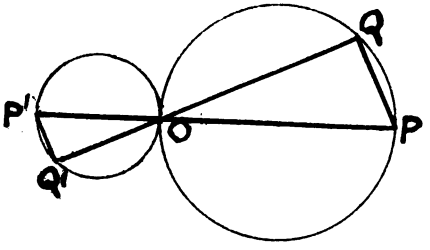


Fig 47





Digitized by Google

Digitized by Google

Fig 48

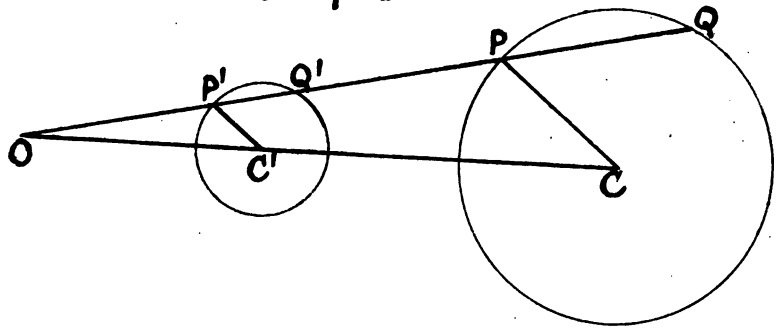


Fig 48

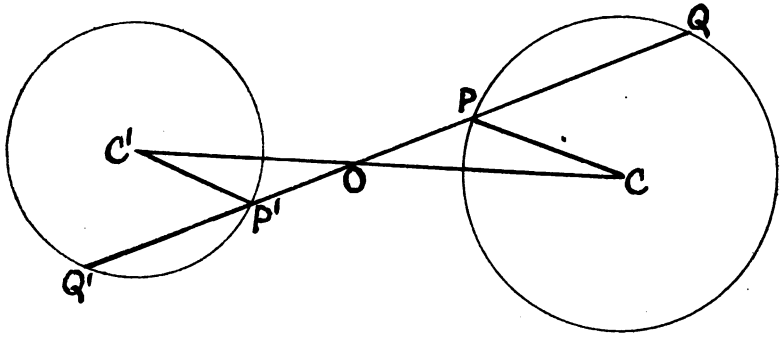
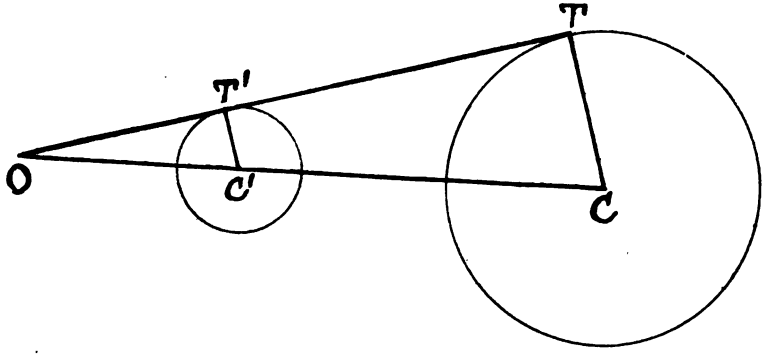


Fig 49



Digitized by Google

Digitized by Google

Digitized by Google

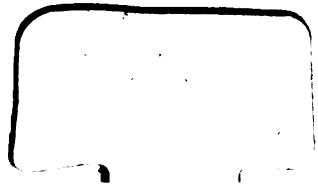
Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google



11-06

Digitized by Google

3 2044